Kinetic of Simple Harmonic Motion

Simple harmonic motion

Simple harmonic motion (SHM)

- is defined as **a periodic motion without loss of energy in which the acceleration of a body is directly proportional to its displacement from the equilibrium position (fixed point) and is directed towards the equilibrium position but in opposite direction of the displacement**.

OR mathematically,

$$a = -\omega^2 x = \frac{d^2 x}{dt^2} \qquad (9.1)$$

where a : acceleration of the body

ω : angular velocity (angular frequency)

x : displacement from the equilibrium position, O

- The **angular frequency**, ω always **constant** thus

$$a \propto -x$$

- The **negative sign** in the equation 9.1 indicates that the **direction of the acceleration, a is always opposite to the direction of the displacement, x.**
- The equilibrium position is a **position** at which the **body** would **come to rest** if it were to **lose all of its energy**.
- Examples of linear SHM system are simple pendulum, horizontal and vertical spring oscillations as shown in Figures 9.1a, 9.1b and 9.1c.

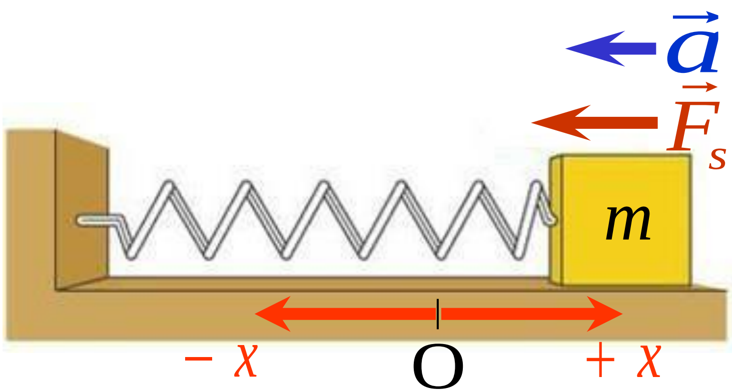


Figure 9.1a

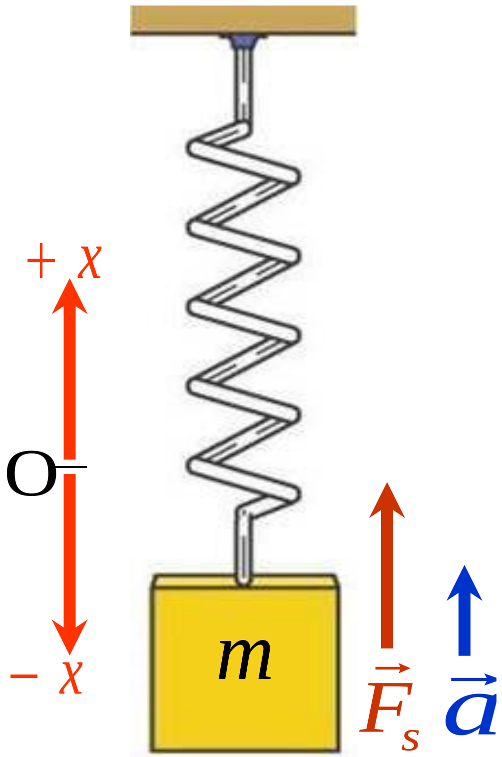


Figure 9.1b

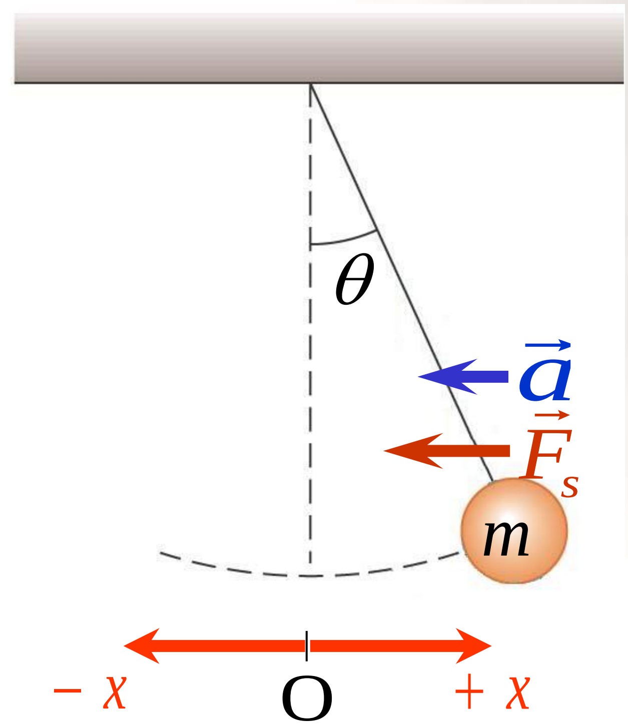


Figure 9.1c

Terminology in SHM

Amplitude (A)

- is defined as the **maximum magnitude of the displacement from the equilibrium position**.
- Its unit is **meter (m)**.

Period (T)

- is defined as **the time taken for one cycle**.
- Its unit is **second (s)**.
- Equation :

$$T = \frac{1}{f}$$

Frequency (f)

- is defined as **the number of cycles in one second**.
- Its unit is **hertz (Hz)** :

Equation : $\omega = 2\pi f$ OR $f = \frac{\omega}{2\pi}$

Equilibrium Position

-- a point where the acceleration of the body undergoing oscillation is zero.

-- At this point, the force exerted on the body is also zero.

Restoring Force

-- the force which causes simple harmonic motion to occcur. This force is proportional to the displacement from equilibrium & always directed towards equilibrium.

$$F_s = -k\,x$$

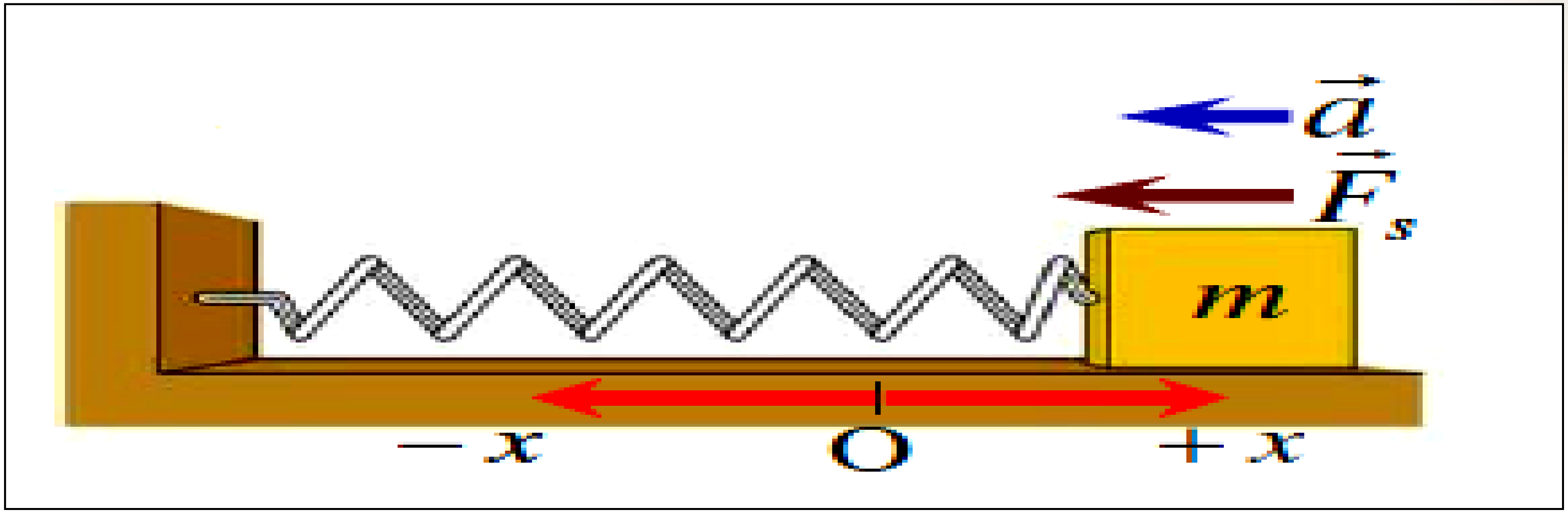
Equation for SHM

-- Consider a system that consists of a block of mass, m attached to the end of a spring with the block free to move on a horizontal, frictionless surface.

-- when the block is displaced to one side of its equilibrium position & released, it moves back & forth repeatedly about a maximum values of displacement x.

-- Maximum value of x is called amplitude, A

-- It can be negative (–) or positive (+).



-- the spring exerts a force that tends to restore the spring to its equilibrium position.

-- Given by Hooke's law :

$$F_s = -k\ x$$

$$F_s = -k\,x$$

-- F_s is known as restoring force.

-- Applying Newton's 2nd Law to the motion of the block :

$$F_{net} = ma$$

$$-k\,x = m\,a$$

$$a = -\frac{k}{m}\,x \qquad \left(\frac{k}{m} : \text{constant value}\right)$$

-- denote ratio k/m with symbol ω^2 :

$$a = -\omega^2\,x$$

[equation for SHM]

-- any system that satisfy this equation is said to exhibit Simple Harmonic Motion (SHM)

This mean that;

$$k = m\,\omega^2$$

$$\omega^2 = k/m$$

Kinematics of SHM

$$x = A\sin(\omega t \pm \pi) \text{ OR } x = -A\sin(\omega t)$$

$$x = A\sin\left(\omega t + \frac{\pi}{2}\right) \text{ OR } x = A\cos(\omega t)$$

$$v_{max} = A\omega$$

$$v = \omega\sqrt{A^2 - x^2}$$

$$a_{max} = A\omega^2$$

Energy in SHM

Potential energy, U

- Consider the oscillation of a spring as a SHM hence the potential energy for the spring is given by

$$U = \frac{1}{2}kx^2 \text{ and } k = m\omega^2$$

$$U = \frac{1}{2}m\omega^2 x^2 \quad \text{(9.13)}$$

- The potential energy in term of time, t is given by

$$U = \frac{1}{2}m\omega^2 x^2 \text{ and } x = A\sin(\omega t + \phi)$$

$$U = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi) \quad \text{(9.14)}$$

Kinetic energy, K

- The kinetic energy of the object in SHM is given by

$$K = \frac{1}{2}mv^2 \text{ and } v = \omega\sqrt{A^2 - x^2}$$

$$K = \frac{1}{2}m\omega^2\left(A^2 - x^2\right) \quad (9.15)$$

- The kinetic energy in term of time, t is given by

$$K = \frac{1}{2}mv^2 \text{ and } v = A\omega\cos(\omega t + \phi)$$

$$K = \frac{1}{2}m\omega^2 A^2\cos^2(\omega t + \phi) \quad (9.16)$$

Total energy, E

- The total energy of a body in SHM is the **sum of its kinetic energy, K and its potential energy, U**.

$$E = K + U$$

- From the principle of conservation of energy, this total energy is always **constant in a closed system** hence

$$E = K + U = \text{constant}$$

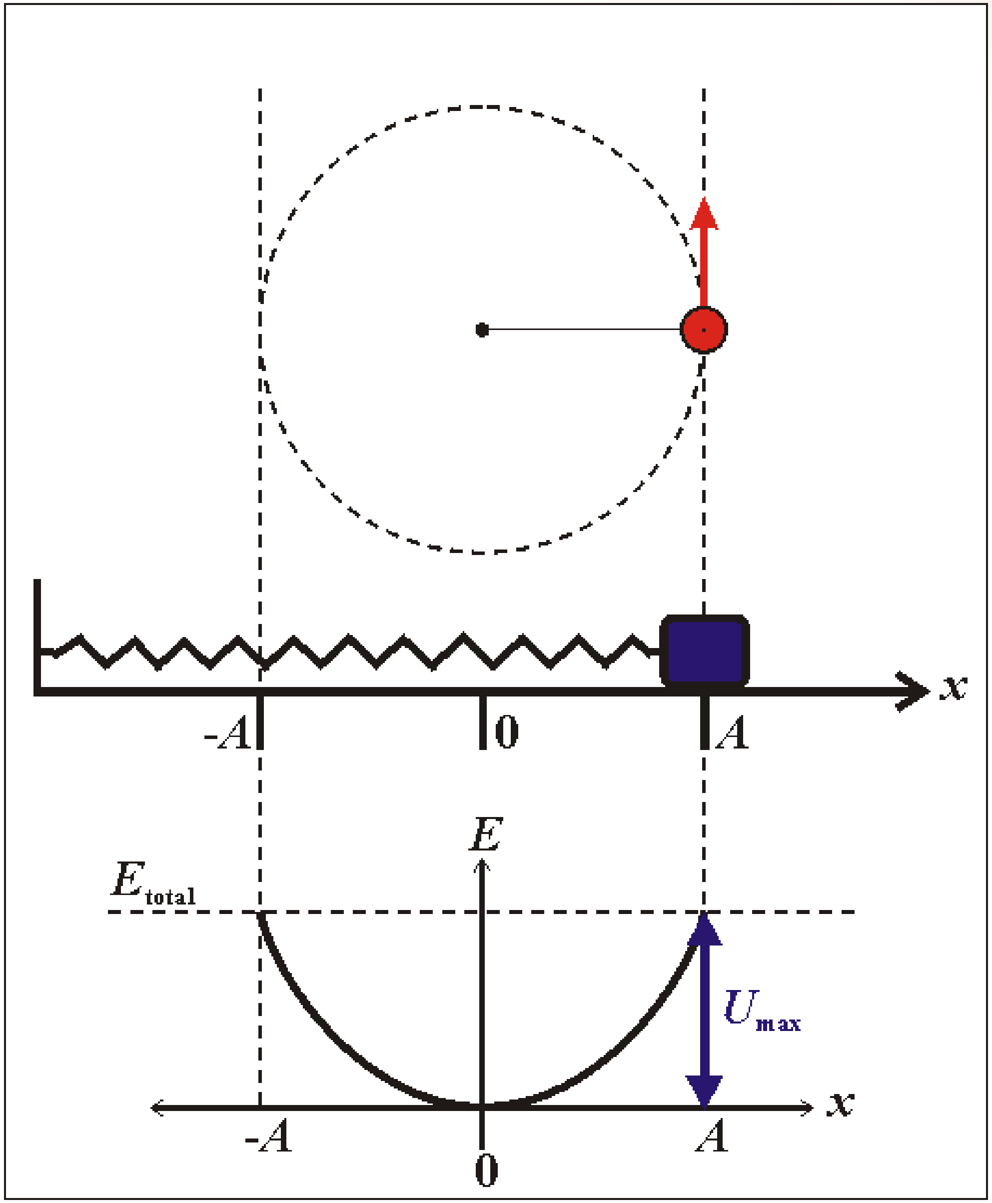
- The equation of total energy in SHM is given by

$$E = \frac{1}{2}m\omega^2\left(A^2 - x^2\right) + \frac{1}{2}m\omega^2 x^2$$

$$E = \frac{1}{2}m\omega^2 A^2 \qquad (9.17)$$

OR

$$E = \frac{1}{2}kA^2 \qquad (9.18)$$



x
-A
0
A
E
E_total
U_max
x
-A
A
0

Example 1 :

An object executes SHM whose displacement x varies with time t according to the relation

$$x = 5.00 \sin\left(2\pi t - \frac{\pi}{2}\right)$$

where x is in centimetres and t is in seconds.

Determine

a. the amplitude, frequency, period and phase constant of the motion,

b. the velocity and acceleration of the object at any time, t ,

c. the displacement, velocity and acceleration of the object at $t = 2.00$ s,

d. the maximum speed and maximum acceleration of the object.

Solution :

a. By comparing

$$x = 5.00 \sin\left(2\pi t - \frac{\pi}{2}\right)$$ with $$x = A \sin(\omega t + \phi)$$

thus

i.

ii.

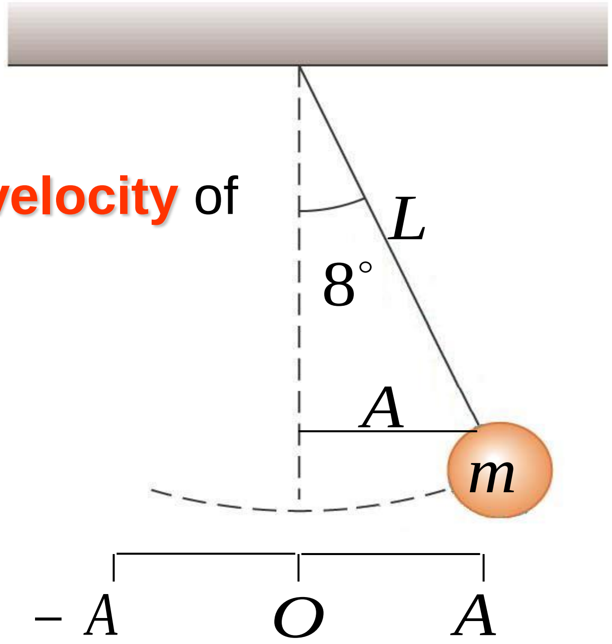
iii. The period of the motion is

iv. The phase constant is

Example 2 :

The length of a simple pendulum is 75.0 cm and it is released at an angle 8° to the vertical. Frequency of the oscillation is 0.576 Hz. Calculate the pendulum's bob speed when it passes through the lowest point of the swing.

(Given $g = 9.81 \text{ m s}^{-2}$)



Solution :

At the **lowest point**, the **velocity** of the pendulum's bob is **maximum** hence

$$L = 0.75 \text{ m}; \theta = 8^\circ$$

Example 3 :

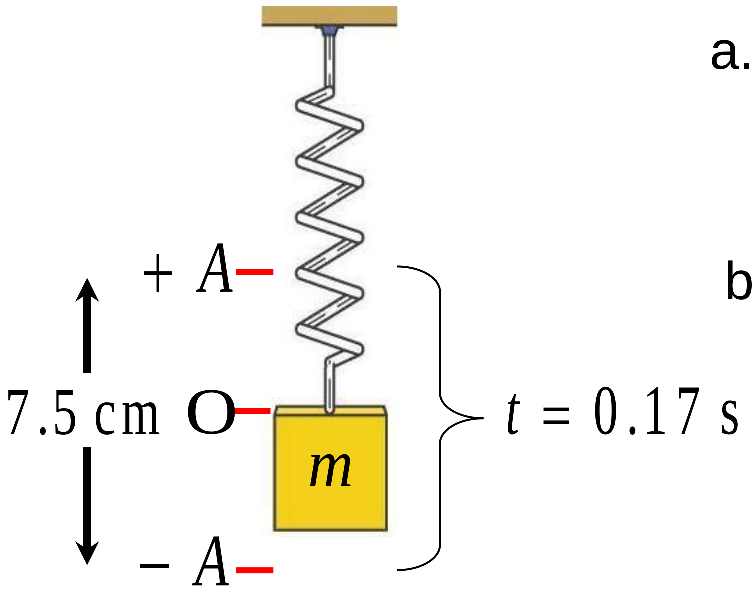
A body hanging from one end of a vertical spring performs vertical SHM. The distance between two points, at which the speed of the body is zero is 7.5 cm. If the time taken for the body to move between the two points is 0.17 s, Determine

a. the amplitude of the motion,

b. the frequency of the motion,

c. the maximum acceleration of body in the motion.

Solution :



a. The amplitude is

$$A = \frac{7.5\times10^{-2}}{2} = 3.75\times10^{-2}\ \text{m}$$

b. The period of the motion is

$$T = 2t = 2(0.17)$$

$$T = 0.34\ \text{s}$$

Example 4 :

An object of mass 450 g oscillates from a vertically hanging light spring once every 0.55 s. The oscillation of the mass-spring is started by being compressed 10 cm from the equilibrium position and released.

a. Write down the equation giving the object's displacement as a function of time.

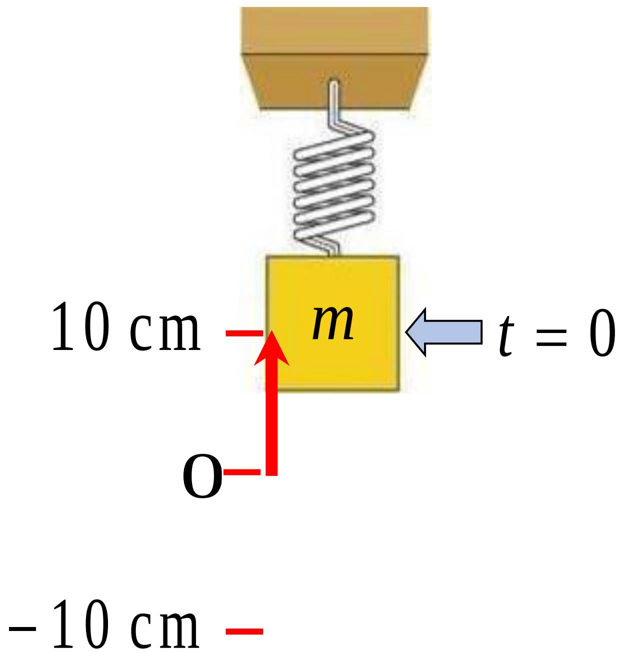
b. How long will the object take to get to the equilibrium position for the first time?

c. Calculate

i. the maximum speed of the object,

ii. the maximum acceleration of the object.

Solution : $m = 0.450 \text{ kg}; T = 0.55 \text{ s}$



a. The amplitude of the motion is $A = 10 \text{ cm}$

The angular frequency of the oscillation is

and the initial phase angle is given by

Therefore the equation of the displacement as a function of time is

$$x = A\sin(\omega t + \phi)$$

OR

Example 5 :

An object of mass 50.0 g is connected to a spring with a force constant of 35.0 N m^{-1} oscillates on a horizontal frictionless surface with an amplitude of 4.00 cm and ω is 26.46 rads^{-1} . Determine

a. the total energy of the system,

b. the speed of the object when the position is 1.00 cm,

c. the kinetic and potential energy when the position is 3.00 cm.

Solution :

$$m = 50.0 \times 10^{-3} \text{ kg}; \; k = 35.0 \text{ N m}^{-1}; \; A = 4.00 \times 10^{-2} \text{ m}$$

a. By applying the equation of the total energy in SHM, thus

Example 6 :

An object of mass 3.0 kg executes linear SHM on a smooth horizontal surface at frequency 10 Hz & with amplitude 5.0 cm. Neglect all resistance forces. Determine :

(a) total energy of the system

(b) The potential & kinetic energy when the displacement of the object is 3.0 cm.

Solution:

Given : $m = 3.0$ kg

$A = 5$ cm $= 0.05$ m

$f = 10$ Hz $\rightarrow$ knowing : $\omega = 2\pi f$

Exercise 1 :

1. A mass which hangs from the end of a vertical helical spring is in SHM of amplitude 2.0 cm. If three complete oscillations take 4.0 s, determine the acceleration of the mass

   a. at the equilibrium position,

   b. when the displacement is maximum.

ANS. : U think ; 44.4 cm s^{-2}

2. A body of mass 2.0 kg moves in simple harmonic motion. The displacement x from the equilibrium position at time t is given by

$$x = 6.0 \sin 2\left(\pi t + \frac{\pi}{6}\right)$$

   where x is in metres and t is in seconds. Determine

   a. the amplitude, period and phase angle of the SHM.

   b. the maximum acceleration of the motion.

   c. the kinetic energy of the body at time $t = 5$ s.

ANS. : 6.0 m, 1.0 s, $\frac{\pi}{3}$ rad ; $24.0\pi^2$ m s^{-2}; 355 J

3. A horizontal plate is vibrating vertically with SHM at a frequency of 20 Hz. What is the amplitude of vibration so that the fine sand on the plate always remain in contact with it?

ANS. : 6.21×10^{-4} m

4. A simple harmonic oscillator has a total energy of E.
   a. Determine the kinetic energy and potential energy when the displacement is one half the amplitude.
   b. For what value of the displacement does the kinetic energy equal to the potential energy?

ANS. : $\frac{3}{4}E, \frac{1}{4}E; \frac{\sqrt{2}}{2}A$

Graphs of SHM

Graph of displacement-time (x-t)

- From the general equation of displacement as a function of time in SHM,
$$x = A\sin(\omega t + \phi)$$
  - If $\phi = 0$, thus $$x = A\sin(\omega t)$$
  - The displacement-time graph is shown in Figure 9.7.

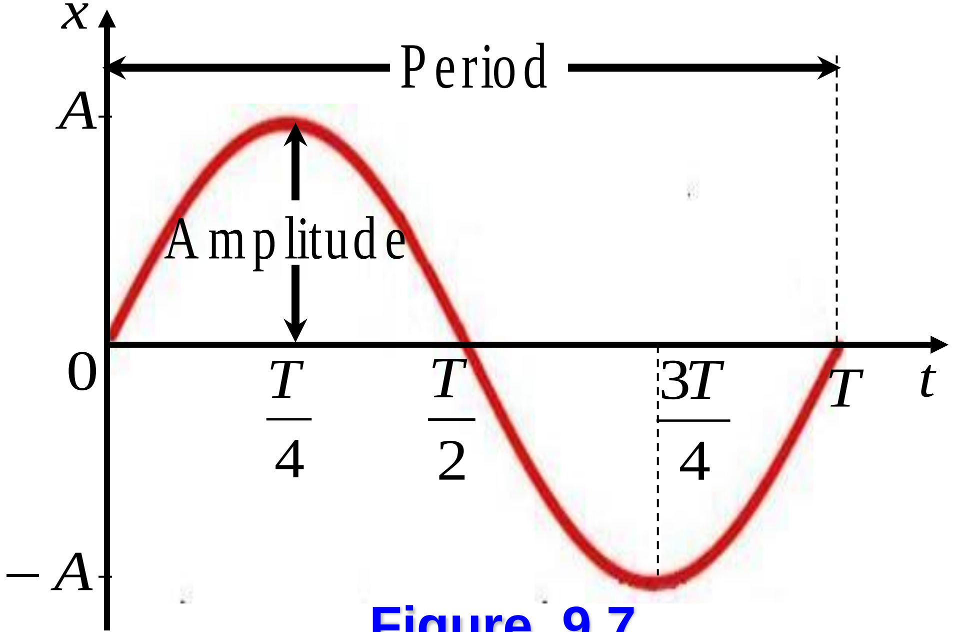


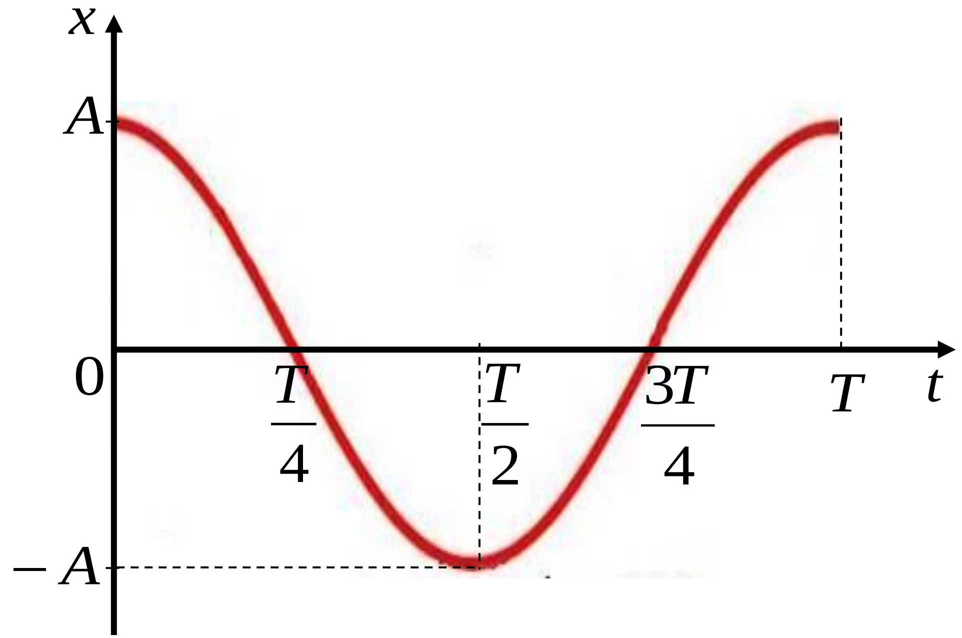
Figure 9.7

- For examples:

a. At $t = 0$ s, $x = +A$

Equation: $x = A\sin\left(\omega t + \frac{\pi}{2}\right)$ OR $x = A\cos(\omega t)$

Graph of x against t:

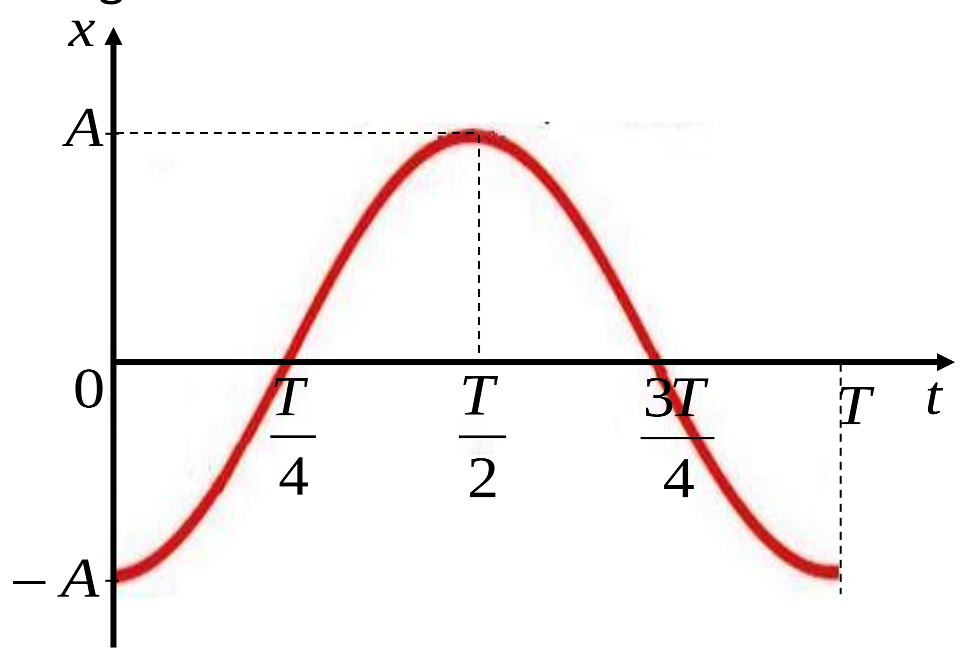


b. At $t = 0$ s, $x = -A$

Equation: $x = A\sin\left(\omega t + \dfrac{3\pi}{2}\right)$ OR $x = A\sin\left(\omega t - \dfrac{\pi}{2}\right)$

OR $x = -A\cos(\omega t)$

Graph of x against t:

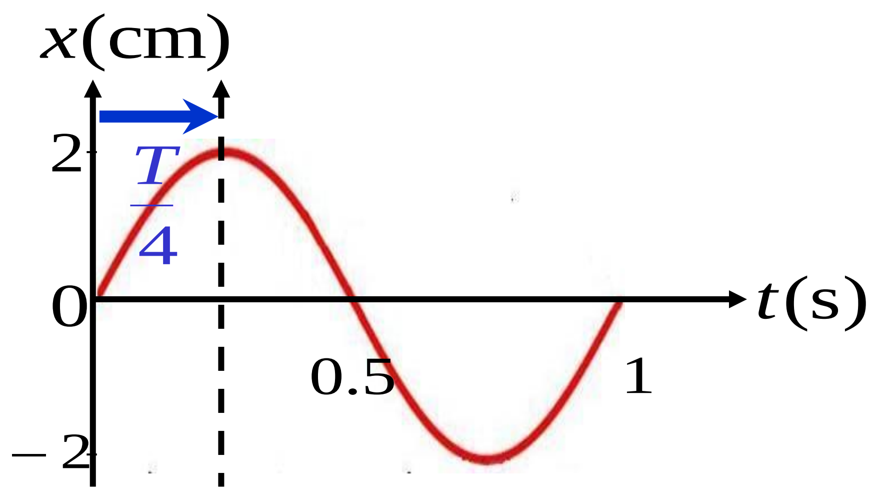


How to sketch the x against t graph when $\phi \neq 0$

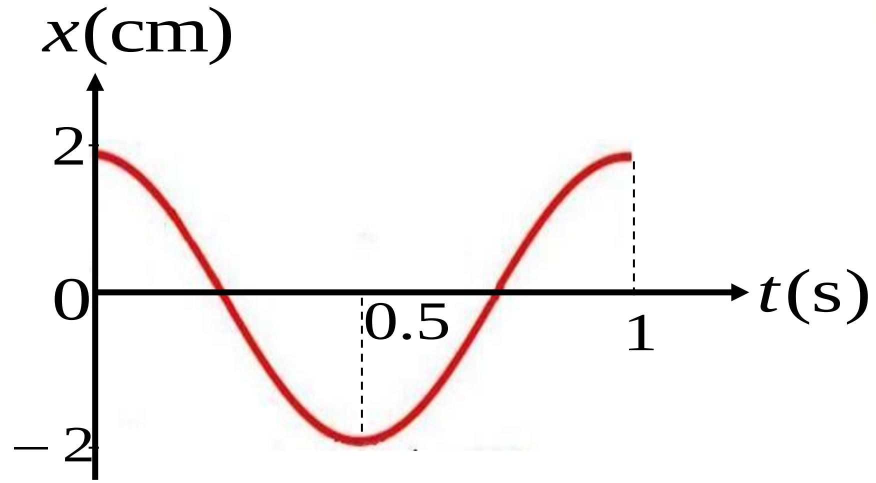
Sketch the x against t graph for the following expression:

$$x = 2\,\text{cm}\sin\left(2\pi t + \frac{\pi}{2}\right)$$

- From the expression,
  - the amplitude, $A = 2\text{ cm}$
  - the angular frequency, $\omega = 2\pi\text{ rad s}^{-1} = \frac{2\pi}{T} \Rightarrow T = 1\text{ s}$
- Sketch the x against t graph for equation $x = 2\sin(2\pi t)$



- Because of $\phi = +\frac{\pi}{2}\ \text{rad} \Rightarrow t = \frac{T}{4}$ hence shift the y-axis to the right by $\frac{T}{4}$
- Sketch the new graph.



RULES

If ϕ = ***negative value***

→shift the y-axis to the left

If ϕ = ***positive value***

→shift the y-axis to the right

Graph of velocity-time (*v-t*)

- From the general equation of velocity as a function of time in SHM,

$$v = A\omega \cos(\omega t + \phi)$$

  - If $\phi = 0$, thus
  - The velocity-time graph is shown in Figure 9.8.

$$v = A\omega \cos(\omega t)$$

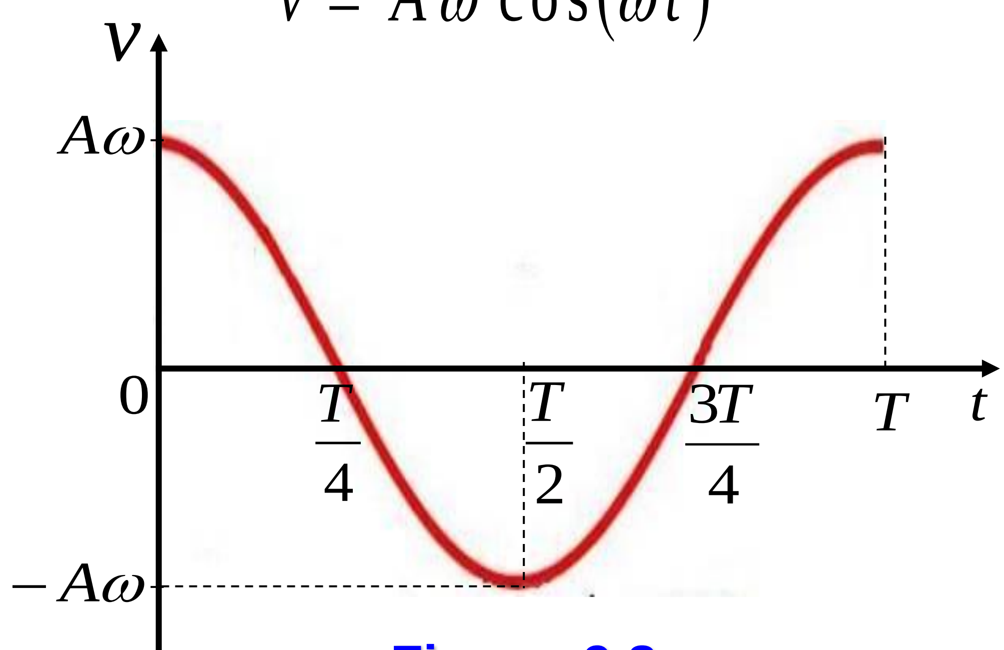


Figure 9.8

Graph of acceleration-time (*a-t*)

- From the general equation of acceleration as a function of time in SHM,
$$a = -A\omega^2 \sin(\omega t + \phi)$$
  - If $\phi = 0$, thus $a = -A\omega^2 \sin(\omega t)$
  - The acceleration-time graph is shown in Figure 9.10.

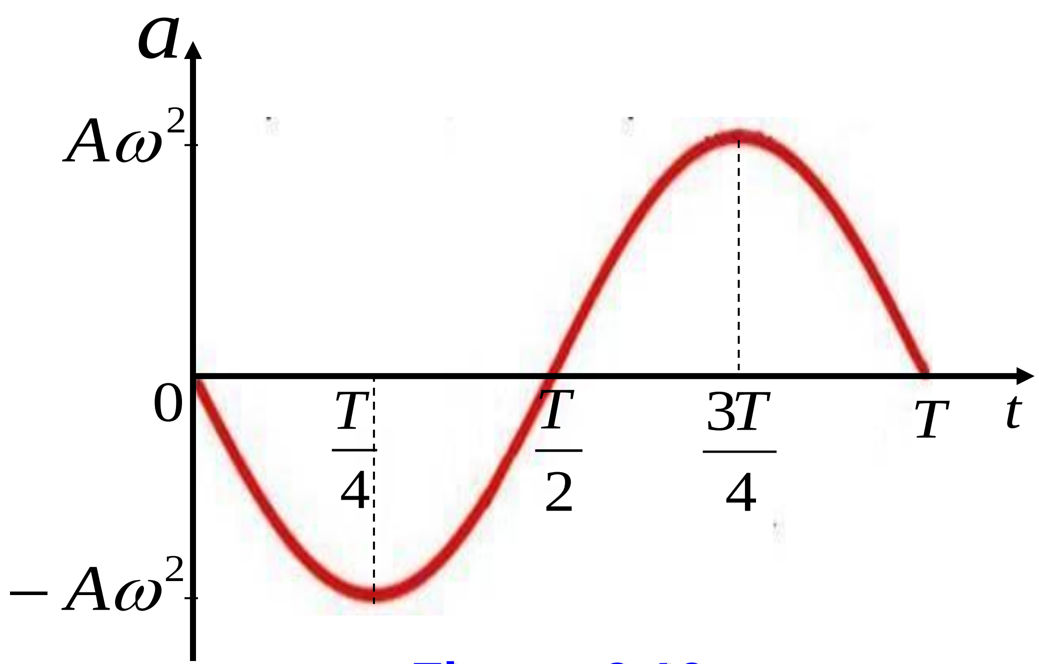


Figure 9.10

- From the relationship between acceleration and displacement,

$$a = -\omega^2 x$$

thus the graph of **acceleration against displacement (*a*-*x*)** is shown in Figure 9.11.

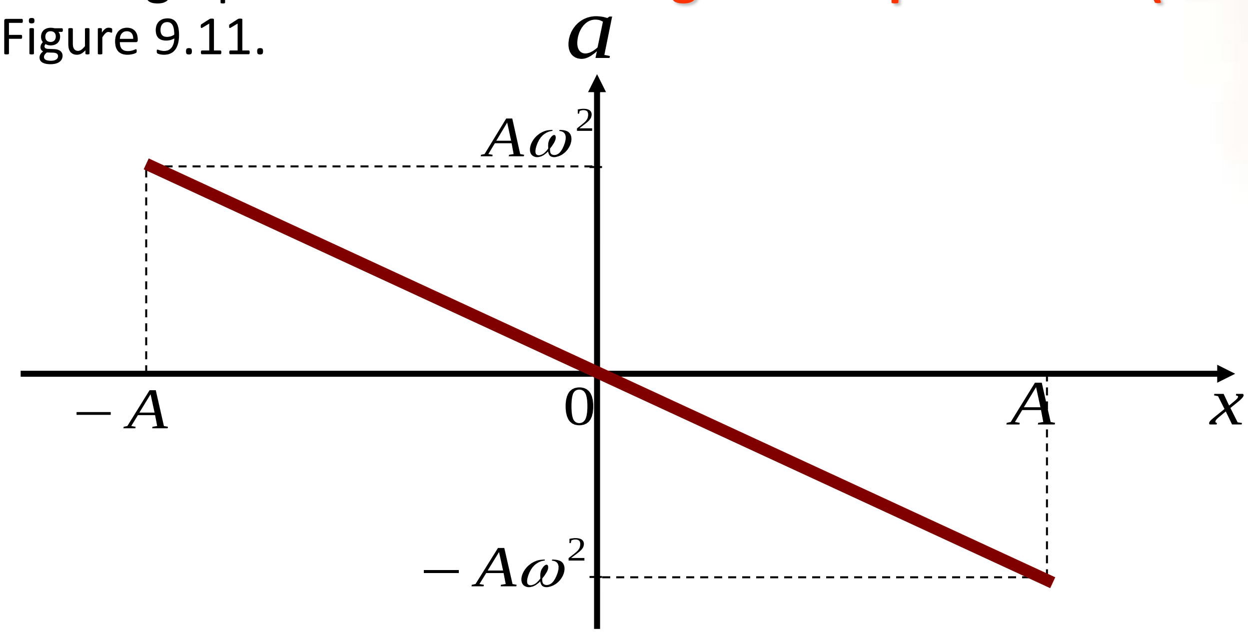


Figure 9.11

- The **gradient of the *a*-*x* graph** represents

$$\text{gradient}, m = -\omega^2$$

Graph of energy-displacement (E-x)

- From the equations of kinetic, potential and total energies as a term of displacement

$$K=\frac{1}{2}m\omega^2\left(A^2-x^2\right);\ U=\frac{1}{2}m\omega^2x^2 \text{ and } E=\frac{1}{2}m\omega^2A^2$$

thus the graph of **energy against displacement (E-x)** is shown in Figure 9.12.

$E=\frac{1}{2}m\omega^2A^2=\text{constant}$

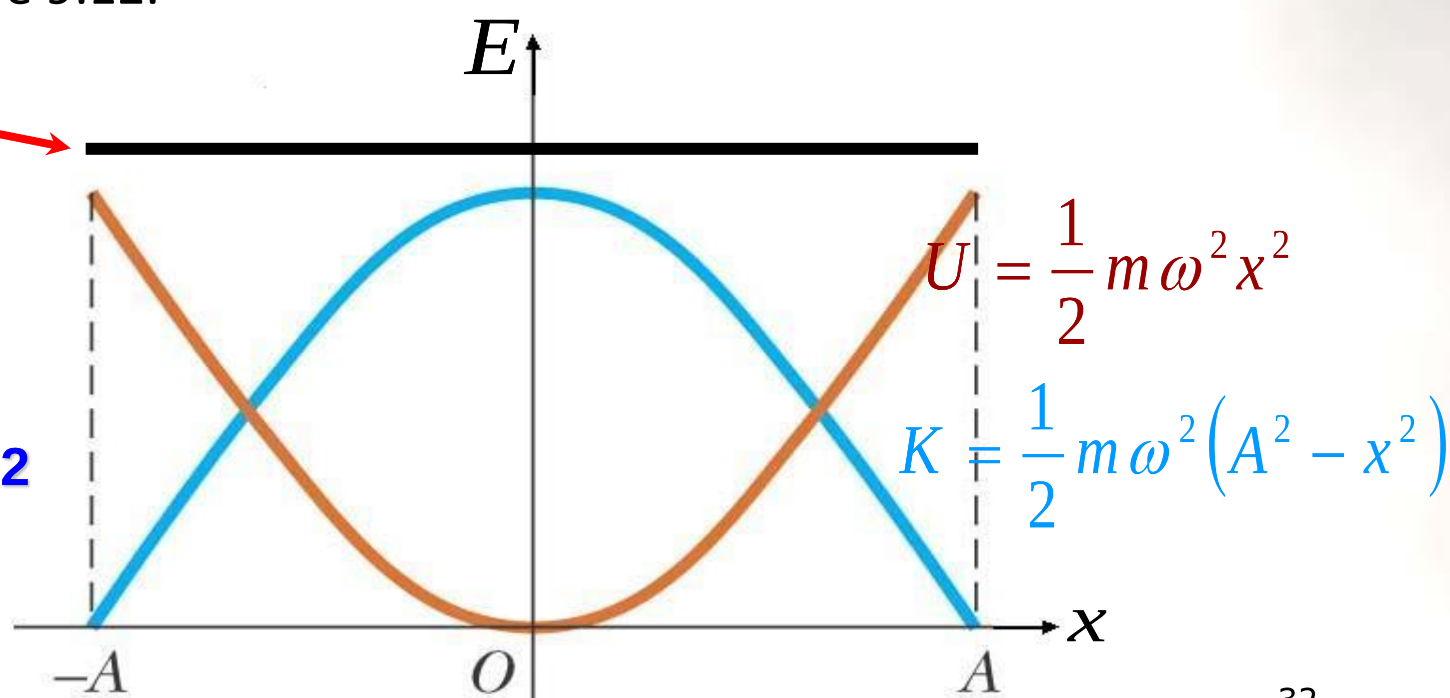


Figure 9.12

- The graph of **Energy against time (E-t)** is shown in Figure 9.13.

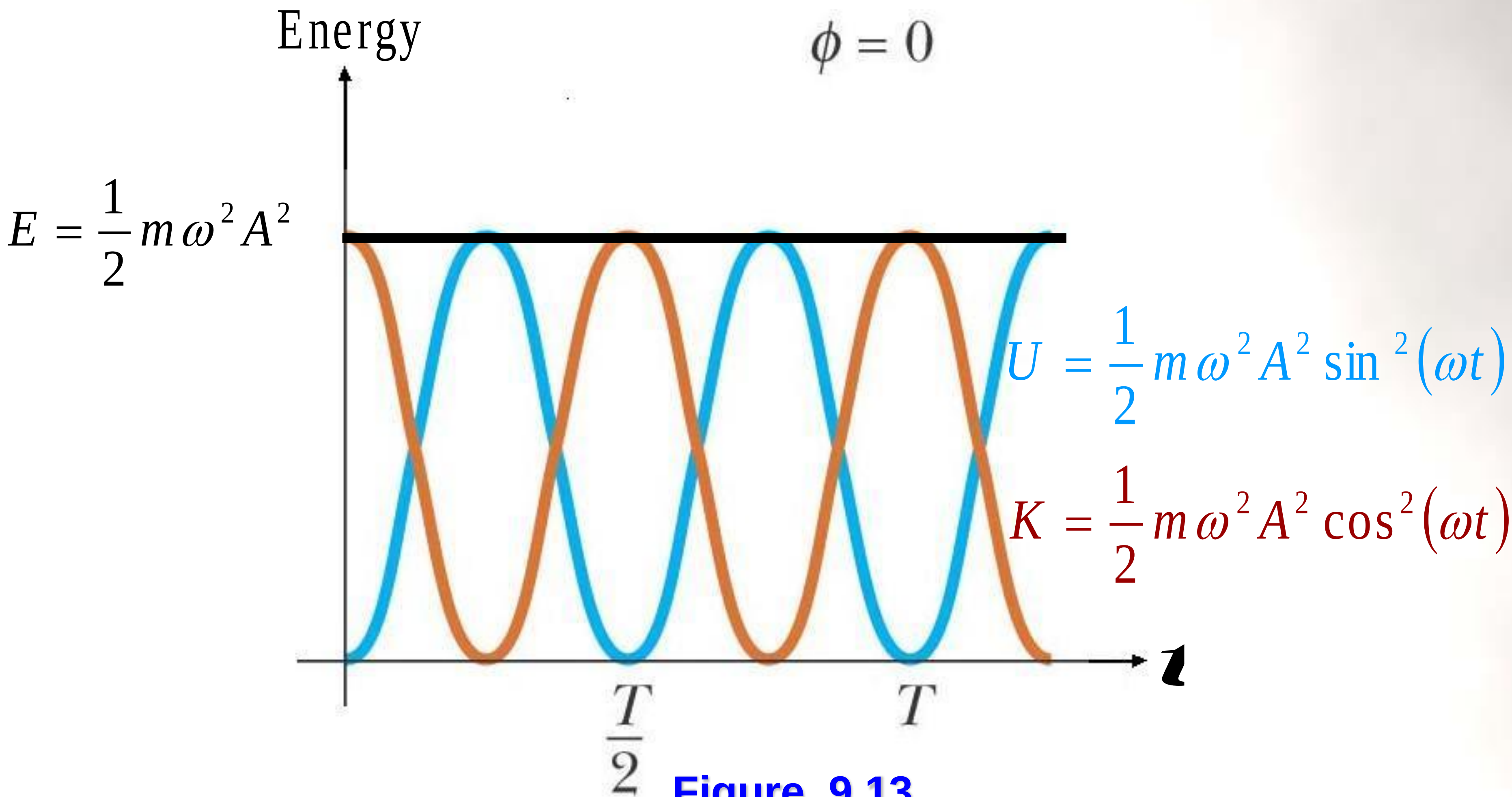


Figure 9.13

Example 7 :

The displacement of an oscillating object as a function of time is shown in Figure 9.14.

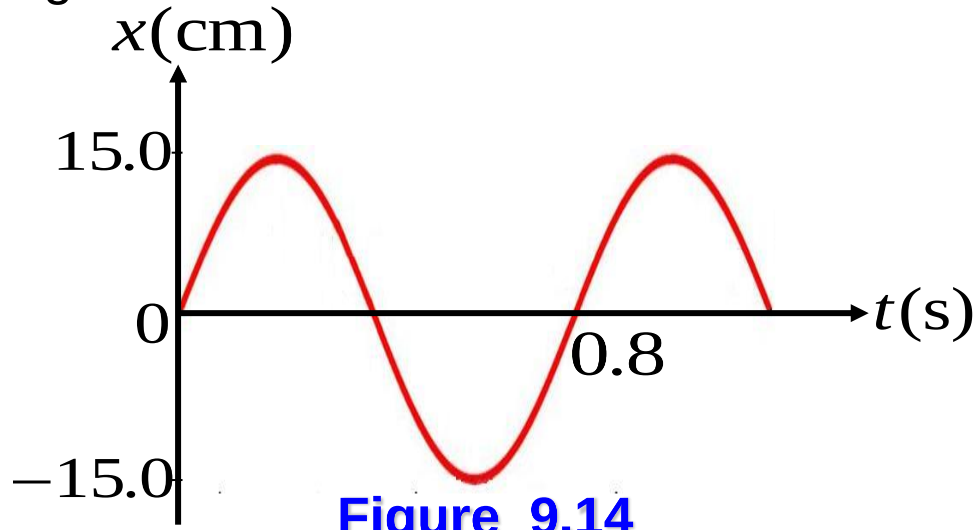


Figure 9.14

From the graph above, determine for these oscillations

a. the amplitude, the period and the frequency,

b. the angular frequency,

c. the equation of displacement as a function of time,

d. the equation of velocity and acceleration as a function of time.

Solution :

a. From the graph,

Amplitude,

Period,

Frequency,

b. The angular frequency of the oscillation is given by

c. From the graph, when t = 0, x = 0 thus $\phi = 0$

By applying the general equation of displacement in SHM

$$x = A\sin(\omega t + \phi)$$

Example 8 :

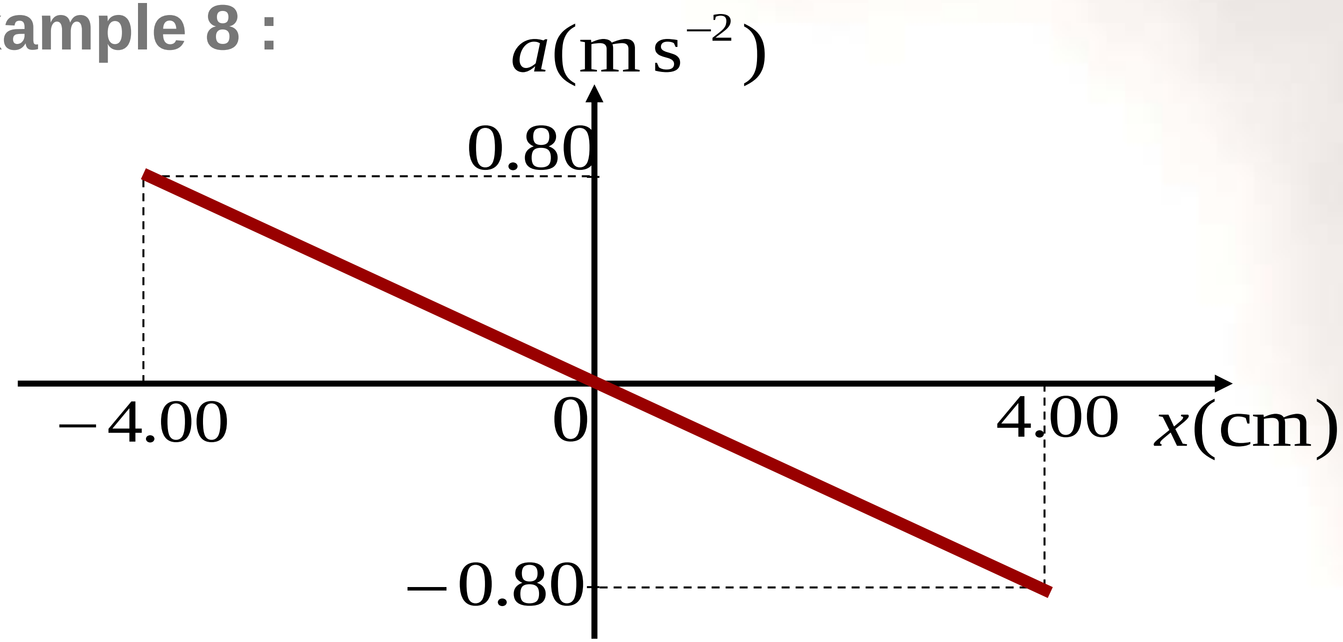


Figure 9.15

Figure 9.15 shows the relationship between the acceleration a and its displacement x from a fixed point for a body of mass 2.50 kg at which executes SHM. Determine

a. the amplitude,

b. the period,

c. the maximum speed of the body,

d. the total energy of the body.

Solution : $m = 2.50\ \text{kg}$

a. The amplitude of the motion is $A = 4.00 \times 10^{-2}\ \text{m}$

b. From the graph, the maximum acceleration is $a_{max} = 0.80\ \text{m s}^{-2}$

By using the equation of maximum acceleration, thus

OR The gradient of the a-x graph is

Example 9 :

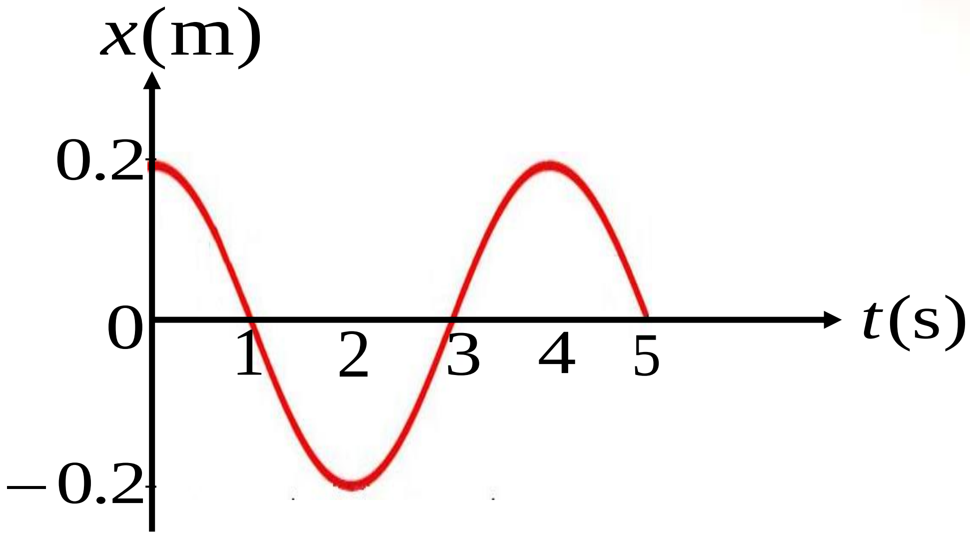


Figure 9.16

Figure 9.16 shows the displacement of an oscillating object of mass 1.30 kg varying with time. The energy of the oscillating object consists the kinetic and potential energies. Calculate

a. the angular frequency of the oscillation,

b. the sum of this two energy.

Solution : $m = 1.30\ \text{kg}$

From the graph,

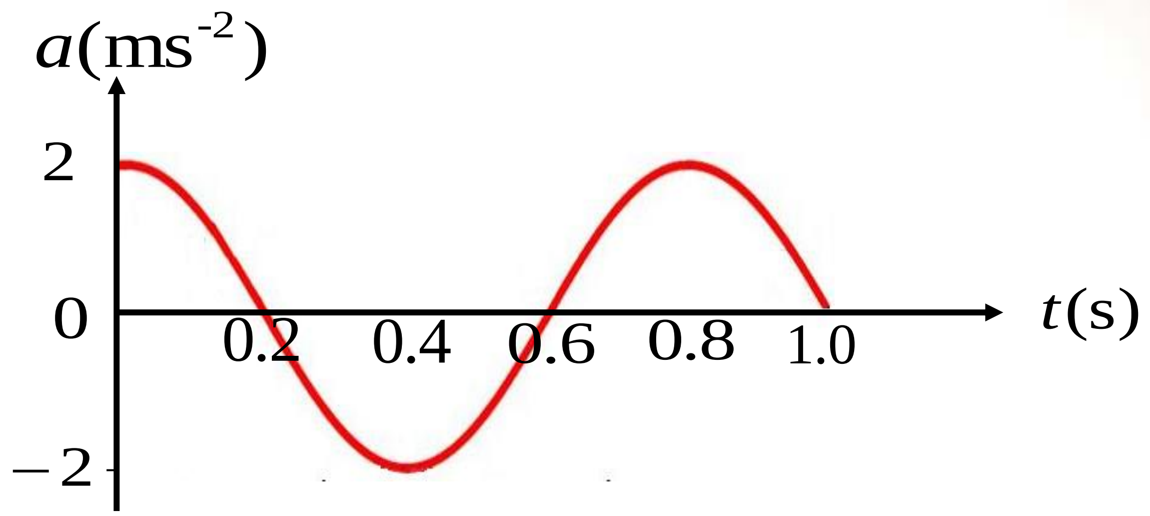
Amplitude,

Period,

a. The angular frequency is given by

b. The sum of the kinetic and potential energies is

Exercise 2 :



1. The graph shows the SHM acceleration-time graph of a 0.5 kg mass attached to a spring on a smooth horizontal surface. By using the graph determine

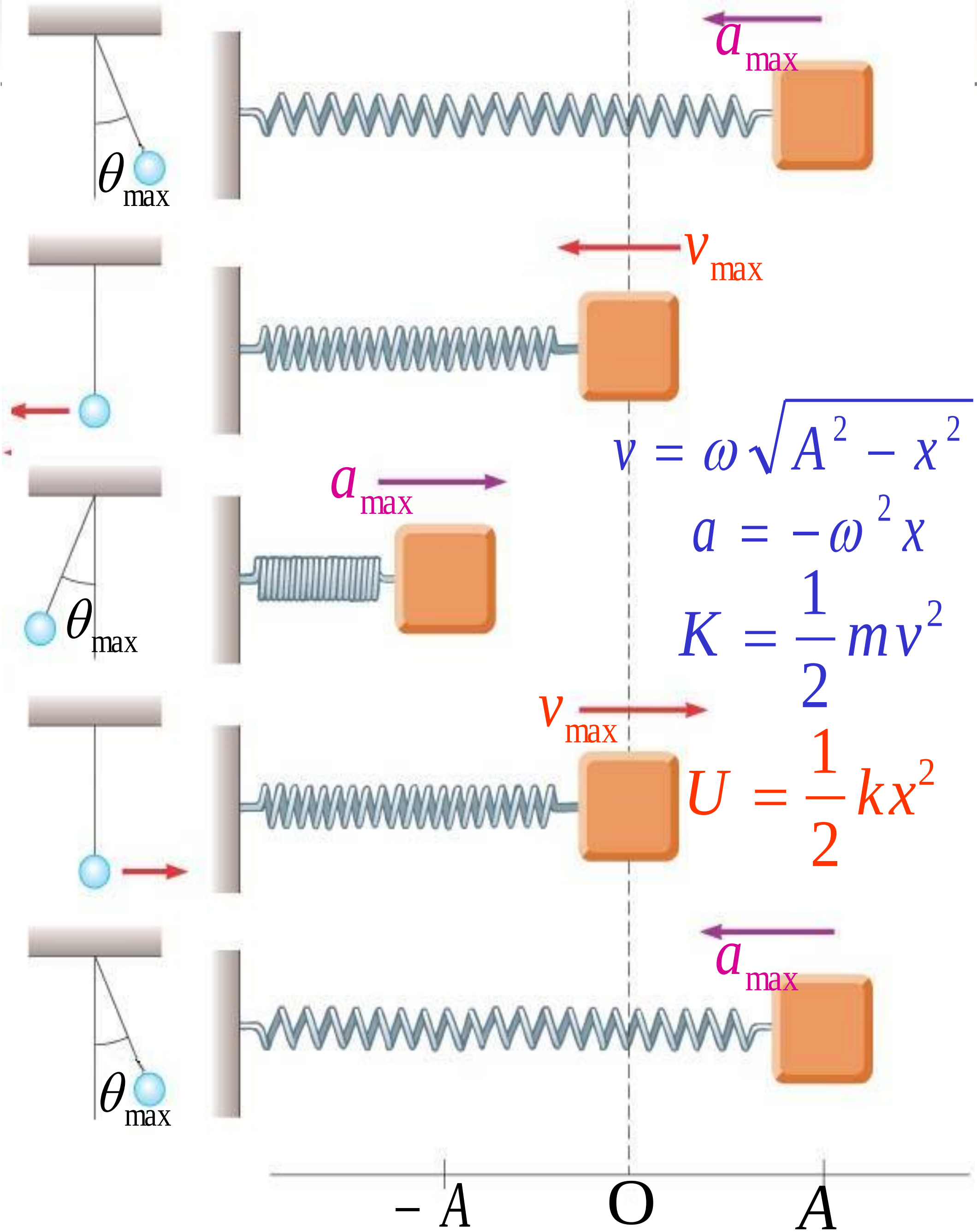
   (a) the spring constant

   (b) the amplitude of oscillation

   (c) the equation of displacement x varies with time, t.

   ANS: 30.8 Nm^{-1}, 0.032 m, $x = 0.032\cos 2.5\pi t$

Summary :



t	x	v	a	K	U
0	A	0	$-A\omega^2$	0	$\frac{1}{2}kA^2$
$\frac{T}{4}$	0	$-A\omega$	0	$\frac{1}{2}mA^2\omega^2$	0
$\frac{T}{2}$	$-A$	0	$A\omega^2$	0	$\frac{1}{2}kA^2$
$\frac{3T}{4}$	0	$A\omega$	0	$\frac{1}{2}mA^2\omega^2$	0
T	A	0	$-A\omega^2$	0	$\frac{1}{2}kA^2$

A. Simple pendulum oscillation

- Figure 9.2 shows the oscillation of the simple pendulum of length, L.

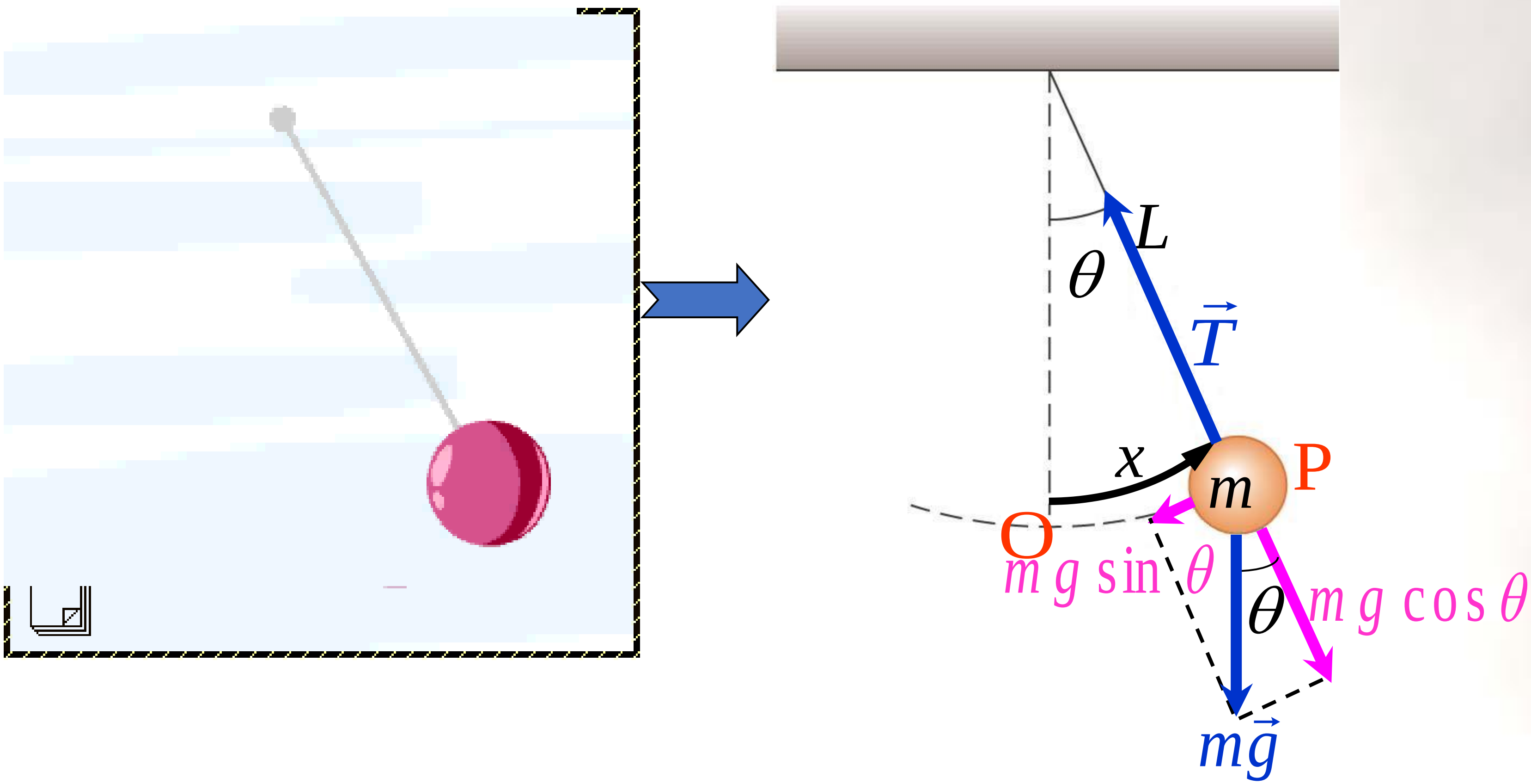
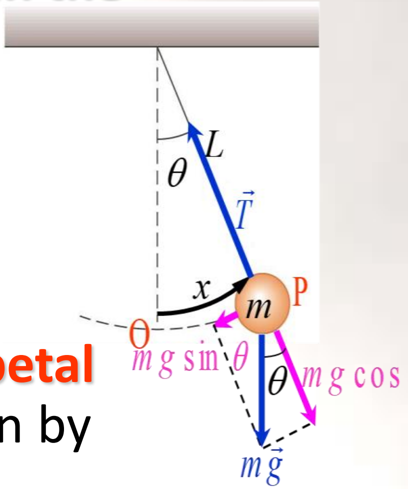


Figure 9.2

- A pendulum bob is pulled slightly to point P.
- The string makes an angle, θ to the vertical and the arc length, x as shown in Figure 9.2.
- The forces act on the bob are mg, **weight** and T, the **tension in the string**.
- Resolve the weight into
  - the **tangential component** : $mg \sin \theta$
  - the **radial component** : $mg \cos \theta$
- The **resultant force** in the **radial** direction provides the **centripetal force** which enables the bob to move along the arc and is given by

$$T - mg\cos\theta = \frac{mv^2}{r}$$

- The **restoring force**, F_s contributed by the **tangential** component of the weight **pulls** the bob back to equilibrium position. Thus



$$F_s = -mg\sin\theta$$

- The **negative sign** shows that the **restoring force**, F_s is always **against the direction of increasing** $\boldsymbol{x}$.

- For **small angle**, θ;
  - $\boldsymbol{\sin\theta \approx \theta}$ **in radian**
  - **arc length**, x of the bob becomes **straight line** (shown in Figure 9.3) then

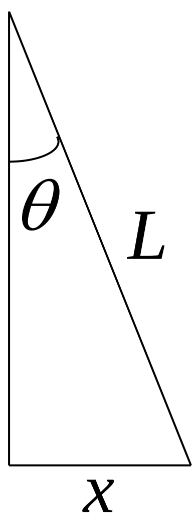


Figure 9.3

$$\sin\theta = \frac{x}{L} \approx \theta$$

$$F_s = -mg\sin\theta$$

thus $$F_s = -mg\left(\frac{x}{L}\right)$$

- By applying Newton's second law of motion,

$$\sum F = ma = F_s$$

$$ma = -\frac{mgx}{L}$$

$$a = -\left(\frac{g}{L}\right)x$$

Thus $a \propto -x$ → **Simple pendulum executes linear SHM**

- By comparing $a = -\left(\frac{g}{L}\right)x$ with $a = -\omega^2 x$

Thus $\omega^2 = \frac{g}{L}$ and $\omega = \frac{2\pi}{T}$

Therefore
$$T = 2\pi\sqrt{\frac{L}{g}} \qquad (9.2)$$

where T : period of the simple pendulum
L : length of the string
g : gravitational acceleration

- The **conditions** for the simple pendulum executes SHM are
  - the angle, θ **has to be small (less than 10°)**.
  - the string has to be **inelastic and light**.
  - only the **gravitational force and tension** in the string acting on the simple pendulum.

B. Spring-mass oscillation

Vertical spring oscillation

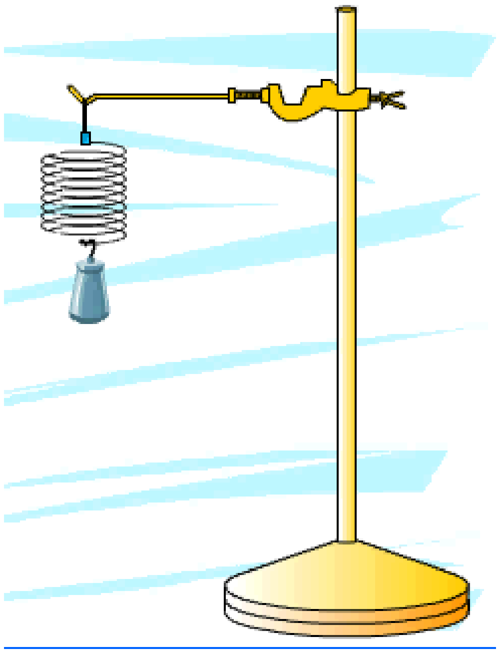


Figure 9.4a

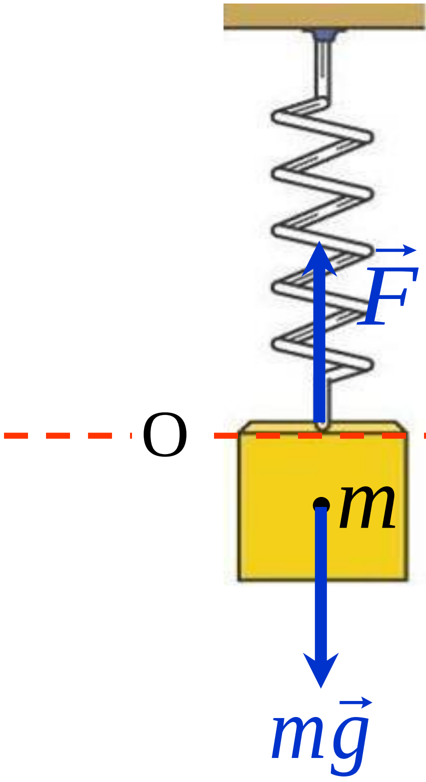


Figure 9.4b

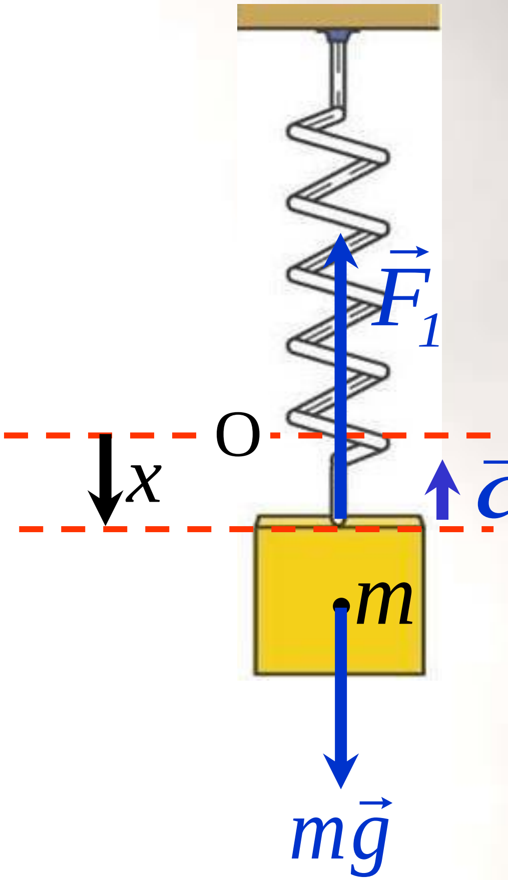


Figure 9.4c

- Figure 9.4a shows a free light spring with spring constant, k hung vertically.
- An object of mass, m is tied to the lower end of the spring as shown in Figure 9.4b. When the object achieves an equilibrium condition, the spring is stretched by an amount x_1 . Thus

$$\sum F = 0 \Rightarrow F - W = 0$$

$$-kx_1 - W = 0 \qquad W = -kx_1$$

- The object is then pulled downwards to a distance, x and released as shown in Figure 9.4c. Hence

$$\sum F = ma$$

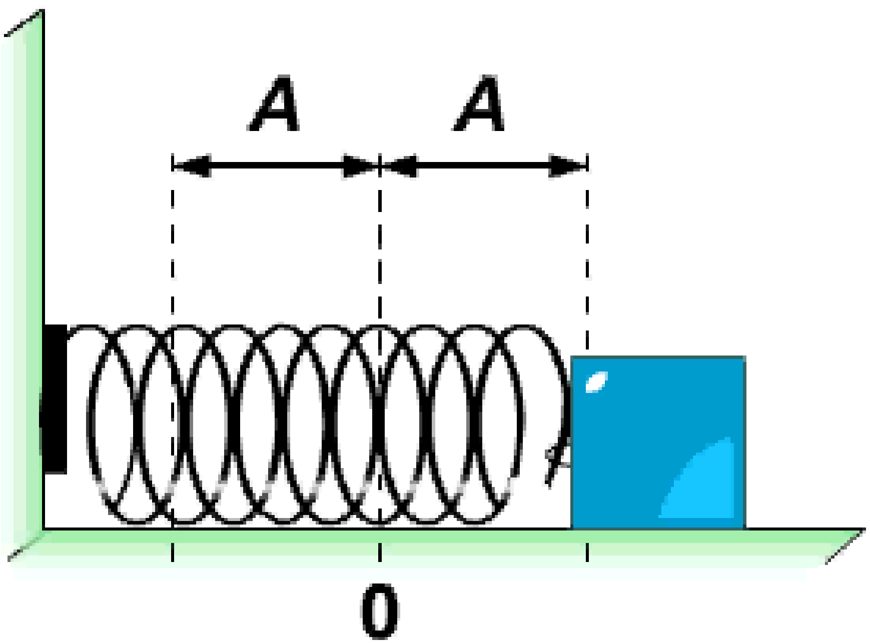
$$F_1 - W = ma \quad \text{and} \quad F_1 = -k(x_1 + x)$$

$$-k(x_1 + x) - (-kx_1) = ma$$

$$a = -\left(\frac{k}{m}\right)x$$

then $a \propto -x$ ⇒ **Vertical spring oscillation executes linear SHM**

Horizontal spring oscillation



$$F_s = ma$$

$$ma = -kx$$

$$a = -\left(\frac{k}{m}\right)x$$

Then

$a \propto -x$ ⇒ **executes linear SHM**

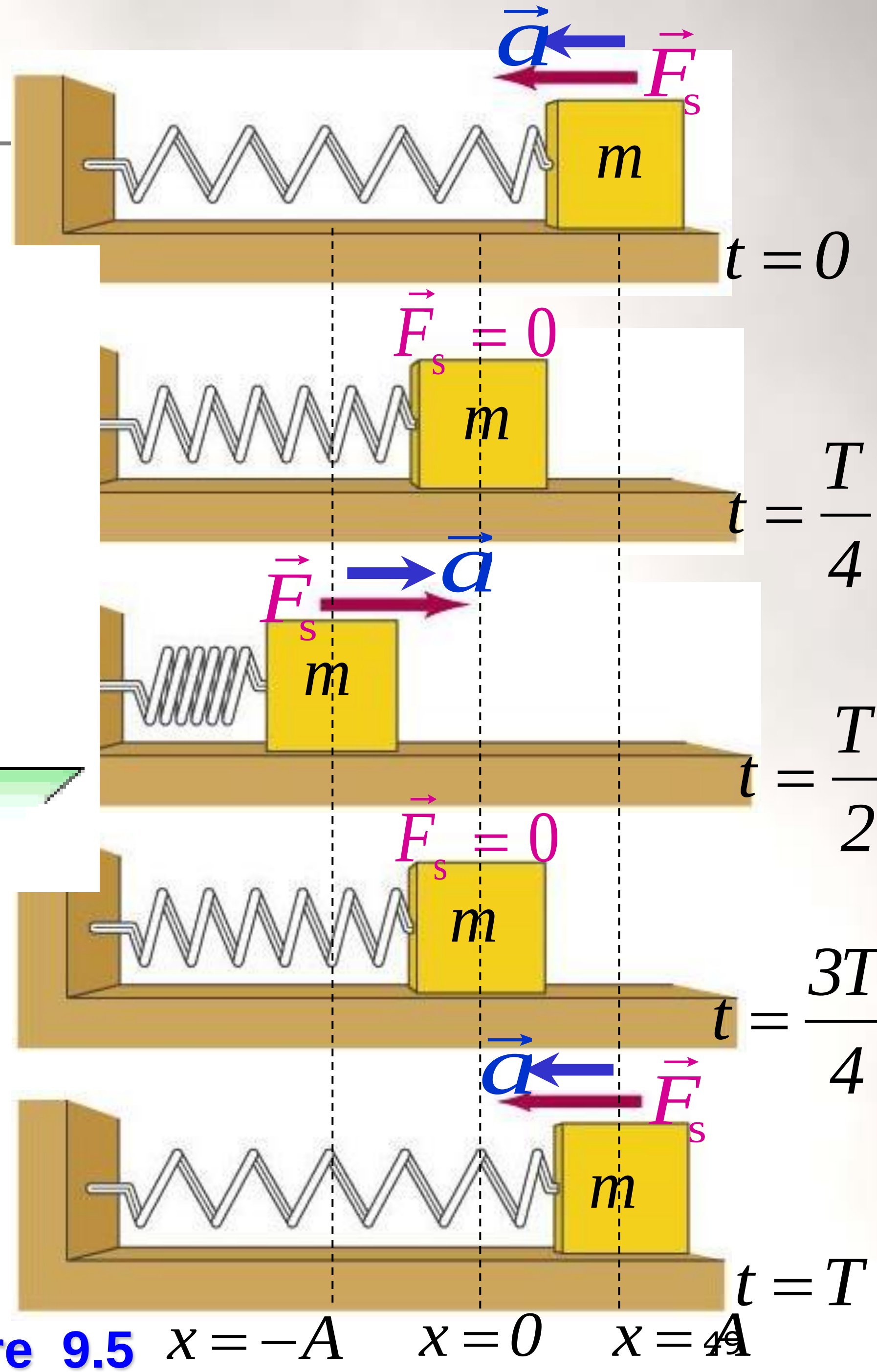


Figure 9.5

- By comparing $a=-\left(\frac{k}{m}\right)x$ with $a=-\omega^2 x$

Thus $\omega^2=\frac{k}{m}$ and $\omega=\frac{2\pi}{T}$

Therefore $T=2\pi\sqrt{\frac{m}{k}}$ (9.3)

where

T : period of the spring oscillation

m : mass of the object

k : spring constant (force constant)

- The **conditions** for the spring-mass system executes SHM are
  - The **elastic limit of the spring is not exceeded when the spring is being pulled**.
  - The spring is **light and obeys Hooke's law**.
  - **No air resistance and surface friction**.

Period of simple harmonic motion

- **Derive and use expression for period of oscillation, *T* for simple pendulum and single spring.**

(i) simple pendulum:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

(ii) single spring:

$$T = 2\pi\sqrt{\frac{m}{k}}$$

Example 10 :

A certain simple pendulum has a period on the Earth surface's of 1.60 s. Determine the period of the simple pendulum on the surface of Mars where its gravitational acceleration is 3.71 m s^{-2}.

(Given the gravitational acceleration on the Earth's surface is g = 9.81 m s^{-2})

Solution : $T_E = 1.60\text{ s};\ g_E = 9.81\text{ m s}^{-2};\ g_M = 3.71\text{ m s}^{-2}$

The period of simple pendulum on the Earth's surface is

But its period on the surface of Mars is given by

Example 11 :

A mass m at the end of a spring vibrates with a frequency of 0.88 Hz. When an additional mass of 1.25 kg is added to the mass m, the frequency is 0.48 Hz. Calculate the value of m.

Solution : $f_1 = 0.88 \text{ Hz}; \; f_2 = 0.48 \text{ Hz}; \; \Delta m = 1.25 \text{ kg}$

The frequency of the spring is given by

After the additional mass is added to the m, the frequency of the spring becomes

Exercise 3 :

1. An object of mass 2.1 kg is executing simple harmonic motion, attached to a spring with spring constant $k = 280$ N m^{-1}. When the object is 0.020 m from its equilibrium position, it is moving with a speed of 0.55 m s^{-1}. Calculate

a. the amplitude of the motion.

b. the maximum velocity attained by the object.

ANS. :5.17×10^{-2} m; 0.597 m s^{-1}

2. The length of a simple pendulum is 75.0 cm and it is released at an angle 8° to the vertical. Calculate

a) the period of the oscillation,

b) the pendulum's bob speed and acceleration when it passes through the lowest point of the swing.

(Given $g = 9.81$ m s^{-2})

ANS.: 1.74s; 0.378ms^{-1}

3. The acceleration of free fall on the Moon is 1/6 the acceleration of free fall on the earth. If the period of a simple pendulum on the earth is 1.0 second, what would its period be on the Moon.

ANS: 2.45 s