CHAPTER 6

Probability Theory

6.1 INTRODUCTION

Probability theory is a mathematical modeling of the phenomemon of chance or randomness. If a coin is tossed in a random manner, it can land heads or tails, but we do not know which of these will occur in a single toss. However, suppose we let s be the number of times heads appears when the coin is tossed n times. As n increases, the ratio $f = s/n$, called the *relative frequency* of the outcome, becomes more stable. If the coin is perfectly balanced, then we expect that the coin will land heads approximately 50 percent of the time, or in other words, the relative frequence will approach 1/2. Alternately, assuming the coin is perfectly balanced, we can arrive at the value 1/2 deductively. That is, any side of the coin is as likely to occur as the other; hence, the chances of getting a heads is one in two, which means the probability of getting a heads is 1/2. Although the specific outcome on any one toss is unknown, the behavior over the long run is determined. This stable long-run behavior of random phenomena forms the basis of probability theory.

A probabilistic mathematical model of random phenonmena is defined by assigning "probabilities" to all the possible outcomes of an experiment. The reliability of our mathematical model for a given experiment depends upon the closeness of the assigned probabilities to the actual limiting relative frequences. This then gives rise to problems of testing and reliability, which form the subject matter of statistics.

6.2 SAMPLE SPACE AND EVENTS

The set S of all possible outcomes of an experiment is called the *sample space*. A particular outcome $a \in S$ is called a *sample point*. An *event* A is a set of outcomes, and so A is a subset of the sample space S. In particular, the set $\{a\}$ consisting of a single sample point $a \in S$ is called an *elementary event*. Furthermore, the empty set Ø and the sample space S are subsets of S and so are events; Ø is called the *impossible event* or the *null event*, and S is called the *sure event*.

Since an event is a set, we can combine events to form new events using the various set operations:

(*i*) $A \cup B$ is the event that occurs iff A occurs *or* B occurs (or both occur).

(*ii*) $A \cap B$ is the event that occurs iff A occurs *and* B occurs.

(*iii*) A^c, the complement of A, also written $\overline{A}$, is the event that occurs iff A does *not* occur.

(As usual in mathematics, iff is short for "if and only if.") Two events A and B are called *mutually exclusive* if they are disjoint, that is, if $A \cap B = \emptyset$. In other words, A and B are mutually exclusive iff they cannot occur simultaneously. Three or more events are mutually exclusive if every two of them are mutually exclusive.

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EXAMPLE 6.1

(a) Experiment: Toss a die and observe the number (of dots) that appears on top.
The sample space S consists of that six possible numbers, namely:

$$S = \{1, 2, 3, 4, 5, 6\}$$

Let A be the event that an even number occurs, B that an odd number occurs, and C that a number greater than 3 occurs. That is, let

$$A = \{2, 4, 6\}, \qquad B = \{1, 3, 5\}, \qquad C = \{4, 5, 6\}$$

Then
$A \cap C = \{4, 6\}$ is the event that a number that is even *and* greater than 3 occurs.
$B \cup C = \{1, 3, 4, 5, 6\}$ is the event that a number that is odd *or* greater than 3 occurs.
$C^c = \{1, 2, 3\}$ is the event that a number that is *not* greater than 3 occurs.
Also, A and B are mutually exclusive: In other words, an even number and an odd number cannot occur simultaneously.

(b) Experiment: Toss a coin three times and observe the sequence of heads (H) and tails (T) that appears.
The sample space S consists of the following eight elements:

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

Let A be the event that two or more heads appear consecutively, and B that all three tosses are the same. That is, let

$$A = \{HHH, HHT, THH\} \quad \text{and} \quad B = \{HHH, TTT\}$$

Then $A \cap B = \{HHH\}$ is the elementary event in which only heads appear. The event that five heads appear is the empty set Ø.

(c) Experiment: Toss a coin until a head appears and then count the number of times the coin is tossed.
The sample space of this experiment is $S = \{1, 2, 3, \ldots\}$. Since every positive integer is an element of S, the sample space is infinite.

Remark: The sample space S in Example 6.1(c), as noted, is not finite. The theory concerning such sample spaces lies beyond the scope of this text. Thus, unless otherwise stated, all our sample spaces S shall be finite.

6.3 FINITE PROBABILITY SPACES

The following definition applies.

Definition: Let S be a finite sample space, say, $S = \{a_1, a_2, \ldots, a_n\}$. Suppose there is assigned to each point $a_i \in S$ a real number p_i satisfying the following properties:

(i) Each p_i is nonnegative; that is, $p_i \geq 0$.
(ii) The sum of the p_i is 1; that is, $p_1 + p_2 + \ldots + p_n = 1$.

Then S is called a *finite probability space*, or *probability model*, and p_i is called the the *probability* of a_i. The *probability* of an event A, written $P(A)$, is defined to be the sum of the probabilities of the points in A. For notational convenience, we write $P(a_i)$ for $P(\{a_i\})$.

EXAMPLE 6.2. Experiment: Let three coins be tossed and the number of heads observed.
[Compare with Example 6.1 (b).]

The sample space is $S = \{0, 1, 2, 3\}$. The following assignments on the elements of S defines a probability space:

$$P(0) = \frac{1}{8}, \qquad P(1) = \frac{3}{8}, \qquad P(2) = \frac{3}{8}, \qquad P(3) = \frac{1}{8}$$

That is, each probability is nonnegative, and the sum of the probabilities is 1. Let A be the event that at least one head appears, and let B be the event that all heads or all tails appear. That is, let

$$A = \{1, 2, 3\} \quad \text{and} \quad B = \{0, 3\}$$

Then, by definition:

$$P(A) = P(1) + P(2) + P(3) = \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$$

and

$$P(B) = P(0) + P(3) = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

Equiprobable Spaces

Frequently, the physical characteristics of an experiment suggest that the various outcomes of the sample space S be assigned equal probabilities. Such a finite probability space S will be called an *equiprobable space*.

Suppose an equiprobable space S has n points. Then the probability of each point must be $1/n$. Moreover, if an event A has r points, then its probability must be $r(1/n) = r/n$. In other words:

$$P(A) = \frac{\text{number of elements in } A}{\text{number of elements in } S} = \frac{n(A)}{n(S)}$$

or

$$P(A) = \frac{\text{number of outcomes favorable to } A}{\text{total number of possible outcomes}}$$

Where $n(A)$ denotes the number of elements in a set A.

Remark: The above formula for $P(A)$ can only be used with respect to an equiprobable space and cannot be used in general.

The expression "at random" will be used only with respect to an equiprobable space. Moreover, the statement "choose a point at random from a set S" means that every point in S has the same probability of being chosen.

EXAMPLE 6.3 Let a card be selected from an ordinary deck of 52 playing cards. Let

$$A = \{\text{the card is a spade}\} \quad \text{and} \quad B = \{\text{the card is a face card}\}$$

(A face card is a jack, queen, or king.) Note that $A \cap B$ is the set of spade face cards. We compute $P(A)$, $P(B)$, and $P(A \cap B)$. Since we have an equiprobable space:

$$P(A) = \frac{\text{number of spades}}{\text{number of cards}} = \frac{13}{52} = \frac{1}{4} \qquad P(B) = \frac{\text{number of face cards}}{\text{number of cards}} = \frac{12}{52} = \frac{3}{13}$$

$$P(A \cap B) = \frac{\text{number of spade face cards}}{\text{number of cards}} = \frac{3}{52}$$

Theorems on Finite Probability Spaces

The following theorem follows directly from the fact that the probability of an event is the sum of the probabilities of its points.

Theorem 6.1: The probability function P defined on the class of all events in a finite probability space S has the following properties:

[P_1] For every event A, $0 \le P(A) \le 1$.

[P_2] $P(S) = 1$.

[P_3] If events A and B are mutually exclusive, then $P(A \cup B) = P(A) + P(B)$.

The next theorem formalizes our intution that if p is the probability that an event E occurs, then $1 - p$ is the probability that E does not occur. (That is, if we hit a target $p = 1/3$ of the times, then we miss the target $1 - p = 2/3$ of the times.)

Theorem 6.2: Let A be any event. Then $P(A^c) = 1 - P(A)$.

The following theorem follows directly from Theorem 6.1.

Theorem 6.3: Let Ø be the empty set, and suppose A and B are any events. Then:

(*i*) $P(\emptyset) = 0$.

(*ii*) $P(A \backslash B) = P(A) - P(A \cap B)$.

(*iii*) If $A \subseteq B$, then $P(A) \leq P(B)$.

Observe that property $[P_3]$ in Theorem 6.1 gives the probability of the union of events in the case that the events are disjoint. The general formula (proved in Problem 6.16) is called the *addition principle*. Specifically:

Theorem 6.4 (Addition Principle): For any events A and B:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

EXAMPLE 6.4. Suppose a student is selected at random from 100 students, where 30 are taking mathematics, 20 are taking chemistry, and 10 are taking mathematics and chemistry. Find the probability p that the student is taking mathematics or chemistry.

Let $M = \{\text{students taking mathematics}\}$ and $C = \{\text{students taking chemistry}\}$. Since the space is equiprobable:

$$P(M) = \frac{30}{100} = \frac{3}{10},\ P(C) = \frac{20}{100} = \frac{1}{5},\ P(M \text{ and } C) = P(M \cap C) = \frac{10}{100} = \frac{1}{10}$$

Thus, by the addition principle (Theorem 6.4):

$$P = P(M \text{ or } C) = P(M \cup C) = P(M) + P(C) - P(M \cap C) = \frac{3}{10} + \frac{1}{5} - \frac{1}{10} = \frac{2}{5}$$

6.4 CONDITIONAL PROBABILITY

Suppose E is an event in a sample space S with $P(E) > 0$. The probability that an event A occurs once E has occured or, specifically, the *conditional probability of A given E*, written $P(A|E)$, is defined as follows:

$$P(A|E) = \frac{P(A \cap E)}{P(E)}$$

As pictured in the Venn diagram in Fig. 6-1, $P(A|E)$ measures, in a certain sense, the relative probability of A with respect to the reduced space E.

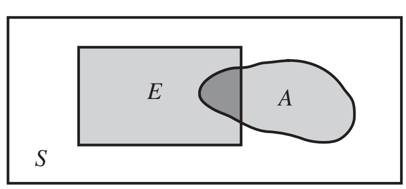

Fig. 6-1

Now suppose S is an equiprobable space, and we let $n(A)$ denote the number of elements in the event A. Then:

$$P(A \cap E) = \frac{n(A \cap E)}{n(S)}, \quad P(E) = \frac{n(E)}{n(S)}, \text{ and so } P(A|E) = \frac{P(A \cap E)}{P(E)} = \frac{n(A \cap E)}{n(E)}$$

We state this result formally:

Theorem 6.5: Suppose S is an equiprobable space and A and B are events. Then:

$$P(A|E) = \frac{\text{number of elements in } A \cap E}{\text{number of elements in } E} = \frac{n(A \cap E)}{n(E)}$$

EXAMPLE 6.5 A pair of fair dice is tossed. The sample space S consists of the 36 ordered pairs (a, b), where a and b can be any of the integers from 1 to 6. (See Problem 6.3.) Thus, the probability of any point is 1/36. Find the probability that one of the die is 2 if the sum is 6. That is, find $P(A|E)$ where

$$E = \{\text{sum is } 6\} \quad \text{and} \quad A = \{2 \text{ appears on at least one die}\}$$

Also find $P(A)$.

Now E consists of five elements, specifically:

$$E = \{(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)\}$$

Two of them, (2, 4) and (4, 2), belong to A. That is:

$$A \cap E = \{\ (2, 4), (4, 2)\ \}$$

By Theorem 6.5, $P(A|E) = 2/5$.

On the other hand, A consists of 11 elements, specifically:

$$A = \{\ (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (1, 2), (3, 2), (4, 2), (5, 2), (6, 2)\ \}$$

also S consists of 36 elements. Hence, $P(A) = 11/36$.

Multiplication Theorem for Conditional Probability

Suppose A and B are events in a sample space S with $P(A) > 0$. By definition of conditional probability:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

Multiplying both sides by $P(A)$ gives us the following useful result:

Theorem 6.6 (Multiplication Theorem for Conditional Probability):

$$P(A \cap B) = P(A)\ P(B|A)$$

The multiplication theorem gives us a formula for the probability that events A and B both occur. It can easily be extended to three or more events $A_1, A_2, \ldots A_m$. That is:

$$P(A_1 \cap A_2 \cap \ldots \cap A_m) = P(A_1) \cdot P(A_2|A_1) \cdot \ldots \cdot P(A_m|A_1 \cap A_2 \cap \ldots \cap A_{m-1})$$

EXAMPLE 6.6 Suppose a lot contains 12 items of which 4 are defective, so 8 are nondefective. Three items are drawn at random one after the other. Find the probability p that all three are nondefective.

The probability that the first item is nondefective is 8/12, since 8 of the 12 items are nondefective. If the first item is nondefective, then the probability that the second item is nondefective is 7/11, since only 7 of the remaining 11 items are nondefective. If the first two items are nondefective, then the probability that the third item is nondefective is 6/10, since only 6 of the remaining 10 items are nondefective. Therefore, by the Multiplication Theorem 6.6,

$$p = \frac{8}{12} \cdot \frac{7}{11} \cdot \frac{6}{10} = \frac{14}{55} \approx 0.25$$

6.7 RANDOM VARIABLES

Let S be a sample space of an experiment. As noted previously, the outcome of the experiment, or the points in S, need not be numbers. However, we frequently wish to assign a specific number to each outcome of the experiment. For example, in the tossing of a pair of die, we may want to assign the sum of the two integers to the outcome. Such an assignment of numerical values is called a *random variable*. More generally, we have the following definition.

Definition: A random variable X is a rule that assigns a numerical value to each outcome in a sample space S.

The *range space* of a random variable X, denoted by R_X, is the set of numbers assigned by X.

Remark: In more formal terminology, X is a function from S to the real numbers $\mathbf{R}$, and $\mathbf{R}_X$ is the range of X. Also, for some infinite sample spaces S, not every function from S to $\mathbf{R}$ is a random variable. However, the sample spaces S here are all finite and so every function from S to $\mathbf{R}$ is a random variable.

EXAMPLE 6.13

(*a*) A pair of fair dice is tossed. (See Problem 6.3.) The sample space S consists of the 36 ordered pairs (a, b), where a and b can be any integers between 1 and 6. That is:

$$S = \{(1, 1), (1, 2), \ldots, (1, 6), (2, 1), \ldots, (6, 6)\}$$

Let X assign to each point in S the sum of the numbers and let Y assign to each point in S the maximum of the two numbers. For example:

$$X(2, 5) = 7, \qquad X(6, 3) = 9, \qquad Y(2, 5) = 5, \qquad Y(6, 3) = 6$$

Then X and Y are random variables on S with respective range spaces

$$R_X = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \quad \text{and} \quad R_Y = \{1, 2, 3, 4, 5, 6\}$$

(*b*) A box contains 12 items of which 3 are defective. A sample of 3 items is selected from the box. The sample space S consists of the $\binom{12}{3} = 220$ different samples of size 3.

Let X denote the number of defective items in the sample; then X is a random variable with range space $R_X = \{0, 1, 2, 3\}$

Probability Distribution of a Random Variable

Let X be a random variable on a finite sample space S with range space $R_X = \{x_1, x_2, \ldots, x_t\}$. The X induces an assignment of probabilities on R_X as follows:

$$p_i = P(x_i) = P(X = x_i) = \text{sum of probabilities of points in } S \text{ whose image is } x_i.$$

The set of ordered pairs $(x_1, p_1), (x_2, p_2, \ldots, (x_t, p_t)$ is usually given by a table as follows:
Such a table is called the *distribution* of the random variable X.

x_i	x_1	x_2	x_3	$\ldots$	x_t
p_i	p_1	p_2	p_3	$\ldots$	p_t

(6.1)

In the case that S is an equiprobable space, we can easily obtain such a distribution as follows:

Theorem 6.10: Let S be an equiprobable space, and let X be a random variable on S with range space

$$R_X = \{x_1, x_2, \ldots, x_t\}$$

Then

$$p_i = P(x_i) = \frac{\text{number of points in } S \text{ whose image is } x_i}{\text{number of points in } S}$$

EXAMPLE 6.14

(*a*) Let X be the random variable in Example 6.13(*a*) which assigns the sum to the toss of a pair of die. Find the distribution of X.
Using Theorem 6.10, we obtain the following:

$P(2) = 1/36$, since there is only one outcome (1, 1) whose sum is 2.

$P(3) = 2/36$, since there are two outcomes, (1, 2) and (2, 1), whose sum is 3.

$P(4) = 3/36$, since there are three outcomes, (1, 3), (2, 2), and (3, 1), whose sum is four.

Similarly, $P(5) = 4/36$, $P(6) = 5/36$, …, $P(12) = 1/36$. Thus, the distribution of X follows:

x_i	2	3	4	5	6	7	8	9	10	11	12
p_i	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

(*b*) Let X be the random variable in Example 6.13(*b*). Find the distribution of X.
Again use Theorem 6.10. We obtain the following:

$$P(0) = \frac{84}{220}, \text{ since there are } \binom{9}{3} = 80 \text{ samples of size 3 with no defective items.}$$

$$P(1) = \frac{108}{220}, \text{ since there are } 3\binom{9}{2} = 108 \text{ samples of size 3 with one defective item.}$$

$$P(2) = \frac{27}{220}, \text{ since there are } 9\binom{3}{2} = 27 \text{ samples of size 3 with two defective items.}$$

$$P(3) = \frac{1}{220}, \text{ since there are is only one sample of size 3 with three defective items.}$$

The distribution of X follows:

x_i	0	1	2	3
p_i	$\frac{84}{220}$	$\frac{108}{220}$	$\frac{27}{220}$	$\frac{1}{220}$

Remark: Let X is a random variable on a probability space $S = \{a_1, a_2, \ldots, a_m\}$, and let $f(x)$ be any polynomial. Then $f(X)$ is the random variable on S defined by

$$f(X)(a_i) = f(X(a_i))$$

That is, $F(X)$ assigns $f(X(a_i))$ to the point a_i in S. Thus, if (11.1) is the distribution of X, then $f(X)$ takes on the values $f(x_1), f(x_2), \ldots, f(x_n)$ with the same corresponding probabilities. Accordingly, the distribution of $F(X)$ consists of the pairs (y_k, q_k) where the probability q_k of y_k is the sum of the p_i's for which $y_k = f(x_i)$.

Expectation of a Random Variable

Let X be a random variable on a probability space $S = \{a_1, a_2, \ldots, a_m\}$. The *mean* or *expectation* of X, denoted by μ, μ_X, or $E(X)$, is defined as follows:

$$\mu = E(X) = X(a_1)P(a_1) + X(a_2)P(a_2) + \ldots + X(a_m)P(a_m) = \Sigma X(a_i)P(a_i)$$

In particular, if X is given by the distribution (6.1), then the *expectation* of X is as follows:

$$\mu = E(X) = x_1p_1 + x_2p_2 + \ldots + x_np_n = \Sigma x_i\, p_i$$

(For notational convenience, we have omitted the limits in the summation symbol Σ.)

EXAMPLE 6.15

(*a*) Suppose a fair coin is tossed six times. The number of heads which can occur with their respective probabilities are as follows:

Then the mean or expectation or expected number of heads follows:

x_i	0	1	2	3	4	5	6
p_i	$\frac{1}{64}$	$\frac{6}{64}$	$\frac{15}{64}$	$\frac{20}{64}$	$\frac{15}{64}$	$\frac{6}{64}$	$\frac{1}{64}$

$$\mu = E(X) = 0\left(\frac{1}{64}\right) + 1\left(\frac{6}{64}\right) + 2\left(\frac{15}{64}\right) + 3\left(\frac{20}{64}\right) + 4\left(\frac{15}{64}\right) + 5\left(\frac{6}{64}\right) + 6\left(\frac{1}{64}\right) = 3$$

This agrees with our intuition. When we toss 6 coins, we expect that 3 should be heads.

(*b*) Consider the random variable X in Example 6.13(*b*) whose distribution appears in Example 6.14(*b*). Then the expectation of X or, in other words, the expected number of defective items in a sample of size 3 follows:

$$\mu = E(X) = 0\left(\frac{84}{220}\right) + 1\left(\frac{108}{220}\right) + 2\left(\frac{27}{220}\right) + 3\left(\frac{1}{220}\right) = 0.75$$

(*c*) Three horses A, B, C are in a race, and suppose their respective probabilities of winning are $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$. Let X denote the payoff function for the winning horse, and suppose X pays \$2, \$6, or \$9 according as A, B, or C wins the race. The expected payoff for the race follows:

$$E(X) = X(A)\ P(A) + X(B)\,P(B) + X(C)\,P(C) = 2\left(\frac{1}{2}\right) + 6\left(\frac{1}{3}\right) + 9\left(\frac{1}{6}\right) = 4.5$$

Variance and Standard Deviation of a Random Variable

Let X be a random variable with mean μ and probability distribution $\{(x_i, y_i)\}$ appearing in (*6.1*). The *variance* Var(X) and *standard deviation* σ of X are defined by:

$$\mathrm{Var}(X) = (x_1 - \mu)^2\, p_1 + (x_2 - \mu)^2 p_2 + \ldots + (x_n - \mu)^2\, p_n = \Sigma\,(x_i - \mu)^2\, p_i = E((X - \mu)^2)$$

$$\sigma = \sqrt{\mathrm{Var}(X)}$$

The following formula is usually more convenient for computing Var(X) than the above:

$$\mathrm{Var}(X) = x_1{}^2\, p_1 + x_2{}^2\, p_2 + \ldots + x_n{}^2 p_n - \mu^2 = \Sigma x_i{}^2 p_i - \mu^2 = E(X^2) - \mu^2$$

Remark: According to the above formula, Var(X) = σ^2.

Both σ^2 and σ measure the weighted spread of the values x_i about the mean μ; however, σ has the same units as μ.

EXAMPLE 6.16

(*a*) Let X denote the number of times heads occurs when a fair coin is tossed six times. The distribution of X appears in Example 6.15(*a*), where its mean $\mu = 3$ is computed. The variance of X is computed as follows:

$$\text{Var}(X) = (0-3)^2\frac{1}{64} + (1-3)^2\frac{6}{64} + (2-3)^2\frac{15}{64} + \cdots + (6-3)\frac{1}{64} = 1.5$$

Alternately:

$$\text{Var}(X) = 0^2\frac{1}{64} + 1^2\frac{6}{64} + 2^2\frac{15}{64} + 3^2\frac{20}{64} + 4^2\frac{15}{64} + 5^2\frac{6}{64} + 6^2\frac{1}{64} = 1.5$$

Thus, the standard deviation is $\sigma = \sqrt{1.5} \approx 1.225$ (heads).

(*b*) Consider the random variable X in Example 6.13(*b*) whose distibution appears in 6.14(*b*) and whose mean $\mu = 0.75$ is computed in Example 6.15(*b*). The variance of X is computed as follows:

$$\text{Var}(X) = 0^2\frac{84}{220} + 1^2\frac{108}{220} + 2^2\frac{27}{220} + 3^2\frac{1}{220} - (.75)^2 = 0.46$$

Thus, the standard deviation of X follows:

$$\sigma = \sqrt{\text{Var}(X)} = \sqrt{.46} = 0.66$$

Binomial Distribution

Consider a binomial experiment $B(n, p)$. That is, $B(n, p)$ consists of n independent repeated trials with two outcomes, success or failure, and p is the probability of success. The number X of k successes is a random variable with the following distribution:

k	0	1	2	$\cdots$	n
$P(k)$	q^n	$\binom{n}{1} q^{n-2} p^2$	$\binom{n}{2} q^{n-2} p^2$	$\cdots$	p^n

The following theorem applies.

Theorem 6.11: Consider the binomial distribution $B(n, p)$. Then:

(*i*) Expected value $E(X) = \mu = np$.

(*ii*) Variance $\text{Var}(X) = \sigma^2 = npq$.

(*iii*) Standard distribution $\sigma = \sqrt{npq}$.

EXAMPLE 6.17

(*a*) The probability that a man hits a target is $p = 1/5$. He fires 100 times. Find the expected number μ of times he will hit the target and the standard deviation σ.

Here $p = 1/5$ and so $q = 4/5$. Hence,

$$\mu = np = 100 \cdot \frac{1}{5} = 20 \quad \text{and} \quad \sigma = \sqrt{npq} = \sqrt{100 \cdot \frac{1}{5} \cdot \frac{4}{5}} = 4$$

(*b*) Find the expected number $E(X)$ of correct answers obtained by guessing in a true-false test with 25 questions, and find the standard deviation σ.

Here $p = 1/2$ and so $q = 1/2$. Hence,

$$E(X) = np = 25 \cdot \frac{1}{2} = 12.5 \quad \text{and} \quad \sigma = \sqrt{npq} = \sqrt{25 \cdot \frac{1}{2} \cdot \frac{1}{2}} = 2.5$$