- Interest can be compounded annually, semiannually, quarterly, monthly, weekly, daily, hourly, by minute, by second, etc. What if it is compounded continuously?

3. Continuous Interest:

When we say that interest is compounded continuously, we mean that k increases without limit in the compound interest formula and the accumulated amount approaches to

$$A = Pe^{rt}$$

When r: annual interest rate

p: Principal

t: number of years

Question: Prove that the Accumulated amount of investing P at r percent annual interest rate compounded continuously is

$$A = Pe^{rt}$$

When r: annual interest rate; p: Principal; t: number of years

Solution:

From the definition of compound interest, the accumulated amount of investing principal p at annual interest rate compounded k times per year is

$$A = P\left(1 + \frac{r}{k}\right)^{kt}$$

When it is compounded continuously, $k \to \infty$. And let $\frac{1}{m} = \frac{r}{k}$

And so $k = rm$ and $m = \frac{k}{r}$

As $k \to \infty,\ m \to \infty$

Thus $A = \lim_{k\to\infty} P\left(1+\frac{r}{k}\right)^{kt} = \lim_{m\to\infty} P\left(1+\frac{1}{m}\right)^{rmt}$

$$= P\left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m}\right)^{rt}$$

We know

$$\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{m} = e$$

Hence, $A = P(e)^{rt}$

Example: Find the amount obtained in one year if \$10,000 is invested at a 6 percent annual interest rate, and interest is compounded

(a) monthly, (b) daily, (c) continuously.

Solution:

In parts (a), and (b), we apply the formula $A = P\left(1+\frac{r}{k}\right)^{kt}$, and in part (c), we use $A = P(e)^{rt}$. In all parts $P = 10{,}000,\ r = 0.06, and\ t = 1$

a) $k = 12,\ A = 10{,}000\left(1+\frac{0.06}{12}\right)^{12} = \$10{,}616.78$

b) $k = 365,\ A = 10{,}000\left(1+\frac{0.06}{365}\right)^{365} = \$10{,}618.31$

c) $A = P(e)^{rt} = 10{,}000(e)^{0.06} = \$10{,}618.37$

Present Value in Continuous interest

$$P = \frac{A}{e^{rt}}$$

Example: What is the present value of an investment whose value in 12 years will be \$20,000 if interest is compounded continuously at an annual rate of 9 percent?

Solution: $A = 20{,}000, \quad t = 12, \quad r = 0.09$

$$P = \frac{A}{e^{rt}} = \frac{20{,}000}{e^{12(0.09)}} = \$6791.91$$

Hence, \$6791.91 invested now will have a value of \$20,000 in 12 years.

An Introduction to Annuity:

An annuity is a series of equal payments made at fixed intervals over a period of time.

For example, A company may decide to set aside a certain amount each year for the purchase of new equipment 10 years from now.

Future Value of an Ordinary Annuity:

Suppose a payment R is made at the end of each payment period and the interest rate per period is i. Then the amount of the annuity after n periods is

$$S = R\frac{(1+i)^n - 1}{i}$$

Example: For a new project, a company decided to invest \$600 every 6 months in an ordinary annuity that pays 8 percent annual interest, compounded semi-annually. What is the value of the annuity in 18 years?

Solution:

Semi-annually: k=2

$$i = \frac{r(\text{annual interest rate})}{k(\text{compounds})} = \frac{0.08}{2} = 0.04, \quad R = 600,$$

$$n = 2 * 18 = 36 \text{ periods}$$

$$S = 600\,\frac{(1+0.04)^{36}-1}{0.04} = \$46{,}558.99$$

Hence, in 18 years the company will have \$46,559 for the new project.

Periodic Payment into a Sinking Fund

To obtain a future value S in an ordinary annuity of n payments at an interest rate i per period, the periodic payment is

$$R = \frac{S}{\frac{(1+i)^n - 1}{i}}$$

Example:

A company wants to have \$1,500,000 to replace old equipment in 10 years. How much must be set aside annually in an ordinary annuity at 7 percent annual interest?

Solution: $\text{Here } i = \frac{0.07}{1} = 0.07, \; n = 10 * 1 = 10, \; S = 1{,}500{,}000$

$$R = \frac{1{,}500{,}000}{\frac{(1+0.07)^{10}-1}{0.07}} = \$108{,}566.25$$

Hence, the company must set aside \$108,566.25 at the end of each year for 10 years.