Polynomials

These lecture notes give a very short introduction to polynomials with real coefficients.

Monomial: A number, a variable or the product of a number and one or more variables.

Polynomial: A monomial or a sum of monomials.

Binomial: A polynomial with exactly two terms.

Trinomial: A polynomial with exactly three terms.

Coefficient: A numerical factor in a term of an algebraic expression.

Degree of a monomial: The sum of the exponents of all of the variables in the monomial.

Degree of a polynomial in one variable: The largest exponent of that variable.

Standard form: When the terms of a polynomial are arranged from the largest exponent to the smallest exponent in decreasing order.

What is the degree of the monomial?

$$5x^4b^2$$

- The degree of a monomial is the sum of the exponents of the variables in the monomial.
- The exponents of each variable are 4 and 2. 4+2 = 6.
  - The degree of the monomial is 6.
  - The monomial can be referred to as a sixth degree monomial.

- A polynomial is a monomial or the sum of monomials

$$4x^2 \qquad 3x^3 - 8 \qquad 5x^2 + 2x - 14$$

- Each monomial in a polynomial is a term of the polynomial.
  - The number factor of a term is called the coefficient.
  - The coefficient of the first term in a polynomial is the ***lead coefficient***.
- A polynomial with two terms is called a ***binomial.***
- A polynomial with three terms is called a ***trinomial.***

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where a_0, a_1, $\ldots$, a_n are real numbers and $n \geq 1$ is a natural number. The domain of a polynomial function is $(-\infty, \infty)$.

The Degree of a *Term* with one variable is the exponent on the variable.

$$5x^2 \Rightarrow 2, \quad 2x^4 \Rightarrow 4, \quad -9m \Rightarrow 1$$

The Degree of a *Term* with more than one variable is the sum of the exponents on the variables.

$$-7x^2y \Rightarrow 3, \quad 2x^4y^2 \Rightarrow 6, \quad -9mn^5z^4 \Rightarrow 10$$

The Degree of a *Polynomial* is the greatest degree of the terms of the polynomial variables.

$$2x^3 - 3x + 7 \Rightarrow 3, \quad 2x^4y^2 + 5x^2y^3 - 6x \Rightarrow 6$$

The degree of a polynomial in one variable is the largest exponent of that variable.

$$2$$

A constant has no variable. It is a 0 degree polynomial.

$$4x+1$$

This is a 1st degree polynomial. 1st degree polynomials are ***linear***.

$$5x^2+2x-14$$

This is a 2nd degree polynomial. 2nd degree polynomials are ***quadratic***.

$$3x^3-8$$

This is a 3rd degree polynomial. 3rd degree polynomials are ***cubic.***

Classify the polynomials by degree and number of terms.

	Polynomial	Degree	Classify by degree	Classify by number of terms
a.	5	Zero	Constant	Monomial
b.	$2x-4$	First	Linear	Binomial
c.	$3x^2+x$	Second	Quadratic	Binomial
d.	x^3-4x^2+1	Third	Cubic	Trinomial

Operations on polynomial

1-Multiplying Two Polynomials

Example 1) : $4t^2(3t^2 + 2t - 5)$

$12t^4 + 8t^3 - 20t^2$

2) $-4m^3(-3m - 6n + 4p)$

$12m^4 + 24m^3n - 16m^3p$

Examples:

$$(x+5)(x^2+10x-3) = x^3+10x^2-3x+5x^2+50x-15$$

$$x^3+15x^2+47x-15$$

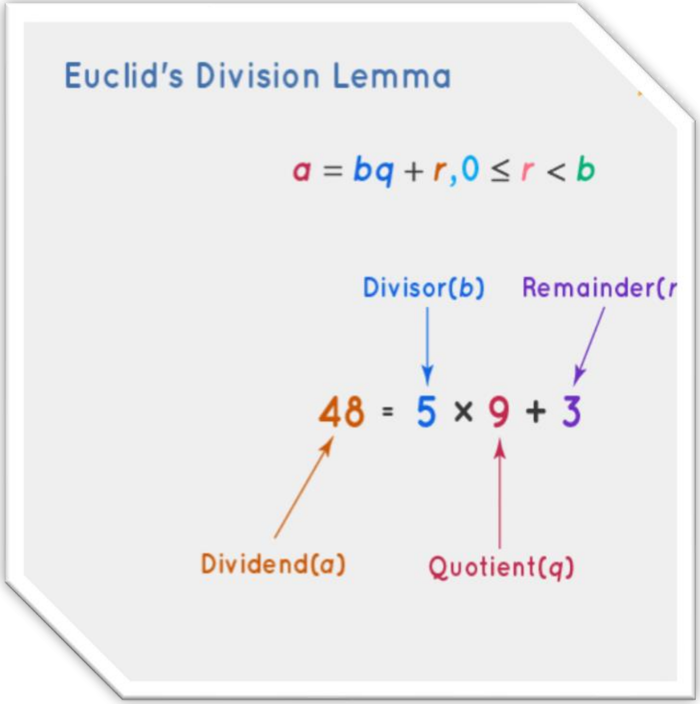
$$(4x^2+x+5)(3x-4) =$$

$$12x^3-16x^2+3x^2-4x+15x-20 =$$

$$12x^3-13x^2+11x-20$$

2-The Division polynomials by polynomial

Euclid's division lemma states that for any two positive integers, say 'a' and 'b', the condition 'a = bq +r', where $0 \leq r < b$ always holds true. Mathematically, we can express this as 'Dividend = (Divisor × Quotient) + Remainder'. A lemma is a statement that is already proved. Euclid, a Greek mathematician, devised Euclid's division lemma.

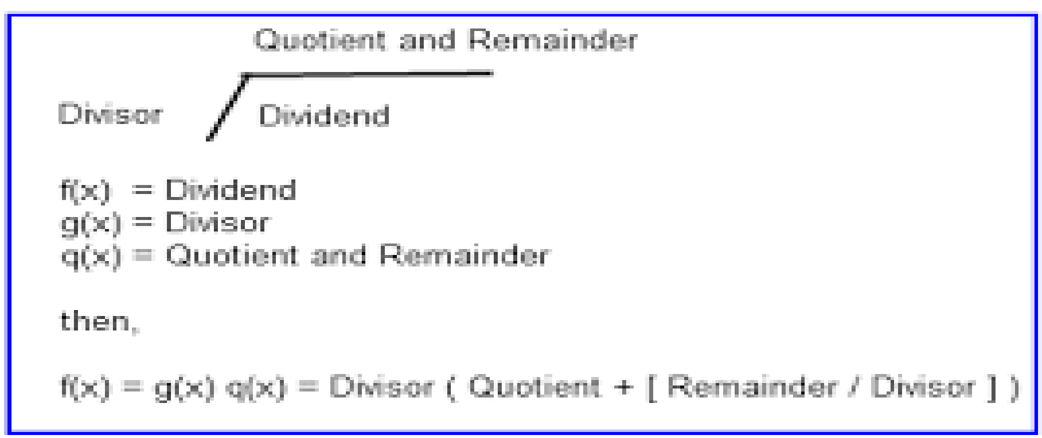


2-The Division polynomials by polynomial

If *f(x)* and *g(x)* are polynomials such that $g(x) \neq 0$, and the degree of *g(x)* is less than or equal to the degree of *f(x)*, there exists a unique polynomials *q(x)* and *r(x)* such that

$$f(x) = g(x)q(x) + r(x)$$

Where $r(x) = 0$ or the degree of *r(x)* is less than the degree of *g(x)*.



Long Division.

use long division to divide polynomials by other polynomials

$$x + 5$$

$$x+3 \overline{)\, x^2 + 8x + 15}$$

$$-x^2 - 3x$$

$$5x + 15$$

$$-5x - 15$$

$$0$$

Check

$$(x+3)(x+5)$$

$$= x^2 + 5x + 3x + 15$$

$$= x^2 + 8x + 15$$

$$\begin{array}{r} x^2 + 2x + 6 \\ x - 1 \overline{) x^3 + x^2 + 4x - 6} \end{array}$$

$$-x^3 + x^2$$

$$0 + 2x^2 + 4x$$

$$-2x^2 + 2x$$

$$-0 + 6x - 6$$

$$-6x + 6$$

$$0$$

1. x goes into x^3? x^2 times.
2. Multiply (x-1) by x^2.
3. Change sign, Add.
4. Bring down 4x.
5. x goes into $2x^2$? 2x times.
6. Multiply (x-1) by 2x.
7. Change sign, Add
8. Bring down -6.
9. x goes into 6x? 6 times.
10. Multiply (x-1) by 6.
11. Change sign, Add .

Divide.

$$\frac{x^3 - 27}{x - 3}$$

$$x - 3 \overline{\smash{\big)}\, x^3 - 27}$$

$$
\begin{array}{r}
x^2 + 3x + 9 \\
x - 3 \overline{\smash{\big)}\, x^3 + 0x^2 + 0x - 27} \\
\underline{-x^3 + 3x^2 } \\
3x^2 + 0x \\
\underline{-3x^2 + 9x } \\
9x - 27 \\
\underline{-9x + 27} \\
0
\end{array}
$$

Long Division.

$$\begin{array}{r} x+2 \\ x-4 \overline{\smash{)}\, x^2-2x-8} \\ -x^2+4x \\ \hline 2x-8 \\ -2x+8 \\ \hline 0 \end{array}$$

Check

$$(x+2)(x-4)$$

$$= x^2-4x+2x-8$$

$$= x^2-2x-8$$

- How to use the Remainder Theorem and the Factor Theorem

Synthetic Division

is a shorthand, or shortcut, method of polynomial division in the special case of dividing by a degree one polynomial -- and it only works in this case.

Divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$

-3	1	0	-10	-2	4
		-3	+9	3	-3
	1	-3	-1	1	1

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x + 3}$$

SYNTHETIC DIVISION: $(5x^3 - 13x^2 + 10x - 8) \div (x - 2)$

STEP #1: Write the Polynomial in DESCENDING ORDER by degree and write any ZERO coefficients for missing degree terms in order

$$\textit{Polynomial Descending Order}: 5x^3 - 13x^2 + 10x - 8$$

STEP #2: Solve the Binomial Divisor = Zero

$$x - 2 = 0; x = 2$$

STEP #3: Write the ZERO-value, then all the COEFFICIENTS of Polynomial.

Coefficients 5 -13 10 -8 = | **Zero = 2**

STEP #4 (Repeat):

(1) ADD Down, (2) MULTIPLY, (3) Product → Next Column

SYNTHETIC DIVISION:

Coefficients	5	-13	10	-8 =	2
	↓	10	-6	8	
	5	-3	4	0 = Remainder	

STEP #5: Last Answer is your REMAINDER

STEP #6: POLYNOMIAL DIVISION QUOTIENT

Write the coefficient "answers" in descending order starting with a Degree ONE LESS THAN Original Degree and include NONZERO REMAINDER OVER DIVISOR at end

(If zero is fraction, then divide coefficients by denominator)

5 -3 4 → $5x^2 - 3x + 4$

$$(5x^3 - 13x^2 + 10x - 8) \div (x - 2) = \boxed{5x^2 - 3x + 4}$$

SYNTHETIC DIVISION: Practice

[1] $(3x^5 - 7x^4 - 4x^2 - 2x - 6)(x-3)^{-1}$

3	–7	0	–4	–2	–6	3
	9	6	18	42	120	
3	2	6	14	40	114	

$$3x^4 + 2x^3 + 6x^2 + 14x + 40 + \frac{114}{x-3}$$

[2] $(8x^4 - 4x^2 + x + 4) \div (2x+1)$

−0.5	8	0	−4	1	4
		−4	2	1	−1
	8	−4	−2	2	3

Divide by 2

$$4x^3 - 2x^2 - x + 1 + \frac{3}{2x+1}$$

[3] $(x^4 - 5x^3 - 13x^2 + 10) \div (x+1)$

[4] $(x^3 + 2x^2 - 5x + 12) \div (x+4)$

REMAINDER THEOREM

The remainder theorem says that if we divide a polynomial $f(x)$ by $x - a$, the remainder is given by $f(a)$

Proof of the Remainder theorem

Let $f(x)$ be a polynomial that is divided by $x - a$

The quotient is another polynomial and the remainder is a constant.

We can write
$$\frac{f(x)}{x-a} \equiv g(x) + \frac{R}{x-a}$$

Multiplying by $x - a$ gives

$$f(x) \equiv (x-a)g(x) + R$$

So,
$$f(a) = (a-a)g(a) + R$$
$$= R$$

$f(x) = 6x^4 - x^3 + 2x^2 - 7x + 2$

$g(x) = 2x + 3$

$$
\begin{array}{r}
3x^3 - 5x^2 + \frac{17}{2}x - \frac{65}{4} \\
2x + 3 \,\big)\, \overline{6x^4 - x^3 + 2x^2 - 7x + 2} \\
+6x^4 + 9x^3 \quad \\
\hline
-10x^3 + 2x^2 - 7x + 2 \\
-10x^3 - 15x^2 \qquad\quad \\
\hline
17x^2 - 7x + 2 \\
+17x^2 + \frac{51}{2}x \qquad \\
\hline
-\frac{65}{2}x + 2 \\
-\frac{65}{2}x - \frac{195}{4} \\
\hline
\frac{203}{4}
\end{array}
$$

$q(x) = 3x^3 - 5x^2 + 17/2\ x - 65/4$

$r = 203/4$

$$f(x) = \left(-\frac{3}{2}\right) = 6\left(-\frac{3}{2}\right)^4 - \left(-\frac{3}{2}\right)^3 + 2\left(-\frac{3}{2}\right)^2 - 7\left(-\frac{3}{2}\right) + 2$$

$$= 6\left(\frac{81}{16}\right) + \frac{27}{8} + 2\left(\frac{9}{4}\right) + \frac{21}{2} + 2$$

$$= \frac{243}{8} + \frac{27}{8} + \frac{9}{2} + \frac{21}{2} + 2$$

$$= \frac{270}{8} + \frac{30}{8} + 2$$

$$= \frac{270 + 120 + 16}{8}$$

$$= \frac{406}{8}$$

$$= \frac{203}{4}$$

:

Given a polynomial function f(*x*): then f(a) equals the remainder of $\dfrac{f(x)}{(x-a)}$

Example: Find the given value

[A] $f(x)=x^3+3x^2-4x-7$, **find** $f(2)$

Method #1: Synthetic Division

1	3	- 4	-7	2
	2	10	12	
1	5	6	5	

Method #2: Substitution/ Evaluate

$f(2)=(2)^3+3(2)^2-4(2)-7$

$f(2)=8+12-8-7$

$f(2)=5$

[B] $f(x)=x^4-5x^2+8x-3$, **find** $f(-3)$

1	0	- 5	8	-3	-3
	-3	9	-12	12	
1	-3	4	-4	9	

$f(-3)=(-3)^4-5(-3)^2+8(-3)-3$

$f(-3)=81-45-24-3=9$

Ex:Find the remainder when $x^3 + 3x^2 - 4x + 1$ is divided by $x - 2$

Solution: Let $f(x) = x^3 + 3x^2 - 4x + 1$

So, $a = 2 \Rightarrow R = f(2)$

$$f(2) = (2)^3 + 3(2)^2 - 4(2) + 1$$

$$= 8 + 12 - 8 + 1$$

$$\Rightarrow \quad R = 13$$

The Factor Theorem:

When f(a)=0 then x−a is a factor of f(x)

or

When x−a is a factor of f(x) then f(a)=0

FACTOR THEOREM:

$(x - \text{a})$ is a factor of $\text{f}(x)$ iff $\text{f}(a) = 0$

remainder = 0

Example: Factor a Polynomial with Factor Theorem

Given a polynomial and one of its factors, find the remaining factors using synthetic division.

$$\textit{Polynomial}: x^3 + 3x^2 - 36x - 108; \quad \textbf{Factor} = (x+3)$$

1	3	-36	-108	-3
↓	-3	0	108	
1	0	-36	0	$= x^2 - 36$

(Synthetic Division)

(x + 6) (x - 6) Remaining factors

$$\textit{Therefore}\ x^3 + 3x^2 - 36x - 108 = (x+3)(x+6)(x-6)$$

Example 1: Find ZEROS/ROOTS of a Polynomial by FACTORING: **(1)** Factor by Grouping **(2)** U-Substitution **(3)** Difference of Squares, Difference of Cubes, Sum of Cubes

[A] $f(x) = x^3 + 2x^2 + 4x + 8$

Factor by Grouping

$$= x^2(x+2) + 4(x+2)$$

$$0 = (x^2+4)(x+2)$$

$$x = \{\pm\ 2i,\ -2\}$$

[B] $f(x) = x^3 - 3x^2 + 9x - 27$

Factor by Grouping

$$= x^2(x-3) + 9(x-3)$$

$$0 = (x^2+9)(x-3)$$

$$x = \{\pm\ 3i,\ 3\}$$

[C] $f(x) = x^4 - 16$

$$= (x^2+4)(x^2-4)$$

$$= (x^2+4)(x+2)(x-2)$$

$$\{\pm 2i,\ \pm 2\}$$

[D] $f(x) = x^3 - 27$

$$= (x-3)(x^2+3x+9)$$

$$\left\{3,\ \frac{-3 \pm 3i\sqrt{3}}{2}\right\}$$

Some facts :-

1-If r is a zero of P(x) then $x - r$ will be a factor of p(x).
2-If $x - r$ is a factor of P(x) then r will be a zero of P(x).

3-If P(x) is a polynomial of degree n and r is a zero of P(x) then P(x) can be written in the following form.
$P(x) = (x - r)q(x),$ where q(x) is a polynomial with degree n−1. q(x) can be found by dividing $p(x) by\, x - r$

C. Solving a Polynomial Equation

Rearrange the terms to have zero on one side:

$$x^2 + 2x = 15 \quad \Rightarrow \quad x^2 + 2x - 15 = 0$$

Factor:

$$(x+5)(x-3) = 0$$

Set each factor equal to zero and solve:

$$(x+5) = 0 \quad \text{and} \quad (x-3) = 0$$

$$x = -5 \qquad\qquad x = 3$$

The only way that $x^2 + 2x - 15$ can $= 0$ is if $x = -5$ or $x = 3$

D. Factors, Roots, Zeros

For our *Polynomial Function*:

$$y = x^2 + 2x - 15$$

The ***Factors*** are: $(x + 5)$ & $(x - 3)$

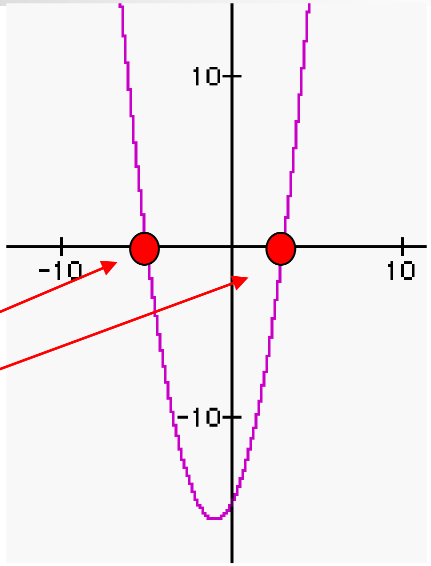
The ***Roots/Solutions*** are: $x = -5$ and 3

The ***Zeros*** are at: $(-5, 0)$ and $(3, 0)$

E. Graph of a Polynomial Function

Here is the graph of our polynomial function:

$$y = x^2 + 2x - 15$$



The *Zeros* of the Polynomial are the values of x when the polynomial equals zero. In other words, the *Zeros* are the x-values where *y equals zero*.

These are also the roots and the x-intercepts.

II. Finding Roots
A. Fundamental Theorem of Algebra

Every Polynomial Equation with a degree higher than zero has at least one root in the set of complex number

Note: If P(x) is a polynomial of degree *n* then P(x) will have exactly n zeroes, some of which may repeat.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdots (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

The Rational Zero Test

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$ has *integer* coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

p = a factor of the constant term a_0

q = a factor of the leading coefficient a_n.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Example 1: List all possible rational zeros given by the Rational Zeros Theorem of $P(x) = 6x^4 + 7x^3 - 4$ (but don't check to see which actually are zeros) .

Solution:

Step 1: First we find all possible values of p, which are all the factors of $a_0 = 4$. Thus, p can be ± 1, ± 2, or ± 4.

Step 2: Next we find all possible values of q, which are all the factors of $a_n = 6$. Thus, q can be ± 1, ± 2, ± 3, or ± 6.

Step 3: Now we find the possible values of $\frac{p}{q}$ by making combinations of the values we found in Step 1 and Step 2. Thus, $\frac{p}{q}$ will be of the form $\frac{\text{factors of 4}}{\text{factors of 6}}$. The possible $\frac{p}{q}$ are

$$\pm\frac{1}{1},\ \pm\frac{2}{1},\ \pm\frac{4}{1},\ \pm\frac{1}{2},\ \pm\frac{2}{2},\ \pm\frac{4}{2},\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{1}{6},\ \pm\frac{2}{6},\pm\frac{4}{6}$$

Step 4: Finally, by simplifying the fractions and eliminating duplicates, we get the following list of possible values for $\frac{p}{q}$.

$$\pm 1,\ \pm 2,\ \pm 4,\ \pm\frac{1}{2},\ \pm\frac{1}{3},\ \pm\frac{2}{3},\ \pm\frac{4}{3},\ \pm\frac{1}{6}$$

Rational Root Theorem

The **rational roots theorem** tells you a list of possible rational roots for a given a polynomial function.

$$\text{Possible Rational Roots} = \frac{\text{factors of the constant}}{\text{factors of the lead coefficient}}$$

Example:

What are the possible rational roots of $6x^3 + 8x^2 - 7x - 3$

The leading coefficient is 6.
The factors of 6 are ±1, ± 2, ±3, ±6.

The constant term is -3.
The factors of -3 are ±1, ±3.

$$\text{Possible Rational Roots} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2, \pm 3, \pm 6}$$

$$= \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{6}, \pm 3, \pm\frac{3}{2}$$

Possible Rational Roots

Ex: $6x^3 + 8x^2 - 7x - 3 = 0$

$$6 \downarrow q \qquad 3 \downarrow p$$

Factors of *p*: $\pm 1, \pm 3$

Factors of *q*: $\pm 1, \pm 2, \pm 3, \pm 6$

Possible zeros: $\pm \frac{p}{q}$

$$\pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \pm \frac{1}{6}, \pm 3, \pm \frac{3}{2}$$

(12 possible zeros)

Find the rational zeros of $f(x) = x^4 - x^3 + x^2 - 3x - 6$.

Solution

Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

By applying synthetic division successively, you can determine that $x = -1$ and $x = 2$ are the only two rational zeros.

$$\begin{array}{r|rrrrr} -1 & 1 & -1 & 1 & -3 & -6 \\ & & -1 & 2 & -3 & 6 \\ \hline & 1 & -2 & 3 & -6 & 0 \end{array}$$

→ 0 remainder, so $x = -1$ is a zero.

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 3 & -6 \\ & & 2 & 0 & 6 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

→ 0 remainder, so $x = 2$ is a zero.

So, $f(x)$ factors as

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

Because the factor $(x^2 + 3)$ produces no real zeros,

$x = -1$ and $x = 2$ are the only *real* zeros of f, Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

The leading coefficient is 2 and the constant term is 3.

$$\textit{Possible rational zeros: } \frac{\text{Factors of 3}}{\text{Factors of 2}} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

Find a fourth-degree polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$f(x) = (x^2 + 2x + 1)(x^2 + 9)$$
$$= x^4 + 2x^3 + 10x^2 + 18x + 9.$$

Find all the zeros of $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$ given that $1 + 3i$ is a zero of f.

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$[x - (1 + 3i)] \quad \text{and} \quad [x - (1 - 3i)]$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r} x^2 - x - 6 \\ x^2 - 2x + 10 \overline{\smash{)}\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\ \underline{x^4 - 2x^3 + 10x^2} \qquad\qquad\quad \\ -x^3 - 4x^2 + 2x \qquad\quad \\ \underline{-x^3 + 2x^2 - 10x} \qquad\quad \\ -6x^2 + 12x - 60 \\ \underline{-6x^2 + 12x - 60} \\ 0 \end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all of its zeros.

Solution

The possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Synthetic division produces the following.

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & 2 & -12 & 8 \\ & & 1 & 1 & 2 & 4 & -8 \\ \hline & 1 & 1 & 2 & 4 & -8 & 0 \end{array}$$

1 is a zero.

$$\begin{array}{r|rrrrr} -2 & 1 & 1 & 2 & 4 & -8 \\ & & -2 & 2 & -8 & 8 \\ \hline & 1 & -1 & 4 & -4 & 0 \end{array}$$

-2 is a zero.

So, you have

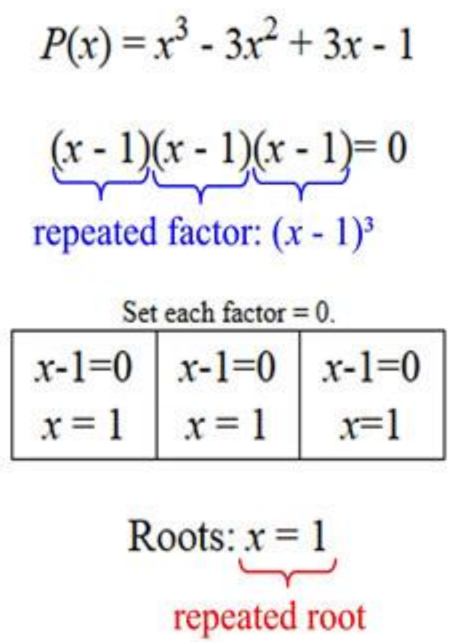
$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x + 2)(x^3 - x^2 + 4x - 4). \end{aligned}$$

You can factor $x^3 - x^2 + 4x - 4$ as $(x - 1)(x^2 + 4)$, and by factoring $x^2 + 4$ as

$$\begin{aligned} x^2 - (-4) &= \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) \\ &= (x - 2i)(x + 2i) \end{aligned}$$

How do you know if a polynomial has a repeated solution?

if both the polynomial and its derivative have a root at r, and that means (x−r) (x−r) are factors of the polynomial. That's why we call it a double root.



Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a **lower bound** for the real zeros of f.

a is a lower bound for the real zeros of f, and b is an upper bound for them $\Leftrightarrow$ **All** the real zeros of f lie in the interval $[a, b]$.

Show that all the real zeros of $f(x) = 4x^3 - 5x^2 - 7x + 2$ must lie in the interval $[-1, 3]$.

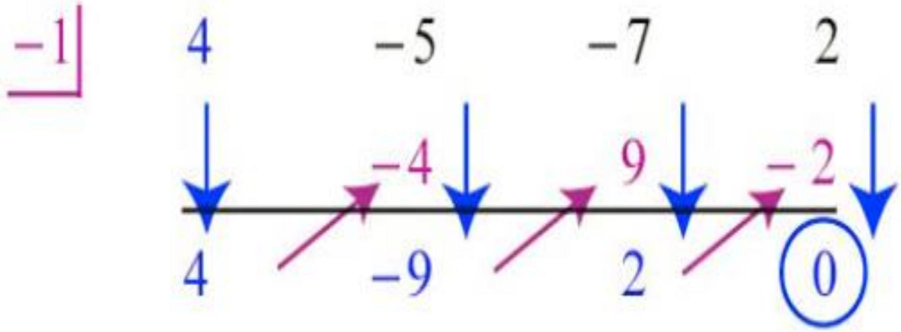
<u>Solution</u>

Use Synthetic Division to divide $f(x)$ by $x - 3$:

3	4	−5	−7	2
		12	21	42
	4	7	14	44

Because $\mathbf{3 > 0}$, and all the entries in the last row are **nonnegative**, 3 an **upper bound** for the real zeros of f.

Use Synthetic Division to divide $f(x)$ by $x-(-1)$:



Because $\mathbf{-1 < 0}$, and the entries in the last row **alternate between nonnegative and nonpositive entries**, -1 is a **lower bound** for the real zeros of f.

1)Find the remainder when the polynomial $p(x) = x^4 + 2x^3 - 4x - 3$ is divided by $(x - 3)$.

2)Determine whether $(2x - 3)$ is a factor of $p(x) = 2x^3 + x^2 + 4x - 15$.

3)List the possible rational roots for the function

$$f(x) = x^4 + 2x^3 - 7x^2 - 8x + 12$$

4)Use the rational root theorem to find linear factorize for the following polynomial function:

$$f(x)=2x^4-11x^3+4x^2+14x-3$$