Polynomials

These lecture notes give a very short introduction to polynomials with real coefficients.

Monomial: A number, a variable or the product of a number and one or more variables.

Polynomial: A monomial or a sum of monomials.

Binomial: A polynomial with exactly two terms.

Trinomial: A polynomial with exactly three terms.

Coefficient: A numerical factor in a term of an algebraic expression.

Degree of a monomial: The sum of the exponents of all of the variables in the monomial.

Degree of a polynomial in one variable: The largest exponent of that variable.

Standard form: When the terms of a polynomial are arranged from the largest exponent to the smallest exponent in decreasing order.

What is the degree of the monomial?

$$5x^4b^2$$

- The degree of a monomial is the sum of the exponents of the variables in the monomial.
- The exponents of each variable are 4 and 2. 4+2 = 6.
  - The degree of the monomial is 6.
  - The monomial can be referred to as a sixth degree monomial.

- A polynomial is a monomial or the sum of monomials

$$4x^2 \qquad 3x^3 - 8 \qquad 5x^2 + 2x - 14$$

- Each monomial in a polynomial is a term of the polynomial.
  - The number factor of a term is called the coefficient.
  - The coefficient of the first term in a polynomial is the ***lead coefficient***.
- A polynomial with two terms is called a ***binomial.***
- A polynomial with three terms is called a ***trinomial.***

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0,$$

where a_0, a_1, $\ldots$, a_n are real numbers and $n \geq 1$ is a natural number. The domain of a polynomial function is $(-\infty, \infty)$.

The Degree of a *Term* with one variable is the exponent on the variable.

$$5x^2 \Rightarrow 2, \quad 2x^4 \Rightarrow 4, \quad -9m \Rightarrow 1$$

The Degree of a *Term* with more than one variable is the sum of the exponents on the variables.

$$-7x^2y \Rightarrow 3, \quad 2x^4y^2 \Rightarrow 6, \quad -9mn^5z^4 \Rightarrow 10$$

The Degree of a *Polynomial* is the greatest degree of the terms of the polynomial variables.

$$2x^3 - 3x + 7 \Rightarrow 3, \quad 2x^4y^2 + 5x^2y^3 - 6x \Rightarrow 6$$

The degree of a polynomial in one variable is the largest exponent of that variable.

2 A constant has no variable. It is a 0 degree polynomial.

$4x+1$ This is a 1st degree polynomial. 1st degree polynomials are ***linear***.

$5x^2+2x-14$ This is a 2nd degree polynomial. 2nd degree polynomials are ***quadratic***.

$3x^3-8$ This is a 3rd degree polynomial. 3rd degree polynomials are ***cubic.***

Classify the polynomials by degree and number of terms.

	Polynomial	Degree	Classify by degree	Classify by number of terms
a.	5	Zero	Constant	Monomial
b.	$2x-4$	First	Linear	Binomial
c.	$3x^2+x$	Second	Quadratic	Binomial
d.	x^3-4x^2+1	Third	Cubic	Trinomial

Operations on polynomial

1-Multiplying Two Polynomials

Example 1) : $4t^2(3t^2 + 2t - 5)$

$12t^4 + 8t^3 - 20t^2$

2) $-4m^3(-3m - 6n + 4p)$

$12m^4 + 24m^3n - 16m^3p$

Examples:

$$\left(x+5\right)\left(x^2+10x-3\right)=x^3+10x^2-3x+5x^2+50x-15$$

$$x^3+15x^2+47x-15$$

$$\left(4x^2+x+5\right)\left(3x-4\right)=$$

$$12x^3-16x^2+3x^2-4x+15x-20=$$

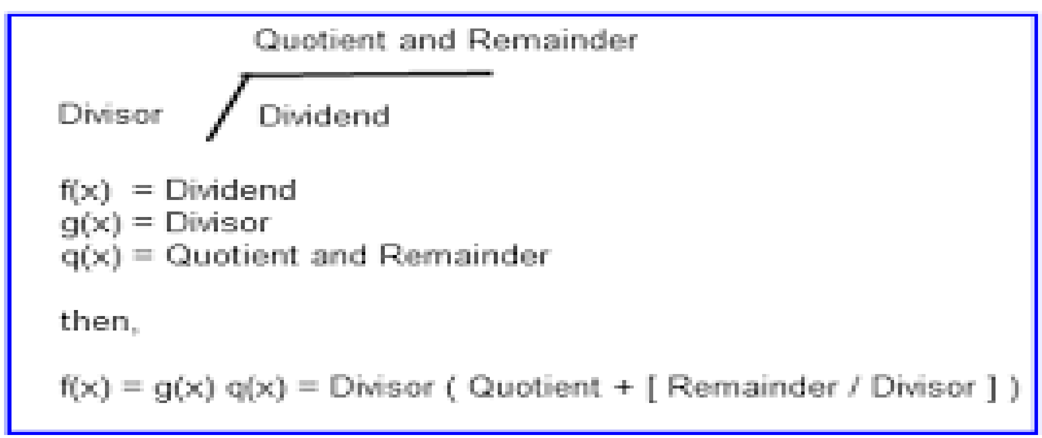
$$12x^3-13x^2+11x-20$$

2-The Division polynomials by polynomial

If $f(x)$ and $g(x)$ are polynomials such that $g(x) \neq 0$, and the degree of $g(x)$ is less than or equal to the degree of $f(x)$, there exists a unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = g(x)q(x) + r(x)$$

Where $r(x) = 0$ or the degree of $r(x)$ is less than the degree of $g(x)$.



Long Division.

use long division to divide polynomials by other polynomials

$$
\begin{array}{r}
x + 5 \\
x+3 \overline{\smash{)}\, x^2 + 8x + 15} \\
-x^2 - 3x \\
\hline
5x + 15 \\
-5x - 15 \\
\hline
0
\end{array}
$$

Check

$$(x+3)(x+5)$$

$$= x^2 + 5x + 3x + 15$$

$$= x^2 + 8x + 15$$

$$
\begin{array}{r}
x^2 + 2x + 6 \\
x-1 \overline{\big) x^3 + x^2 + 4x - 6} \\
\boxed{-x^3 + x^2} \qquad\qquad \\
\hline
0 + 2x^2 + 4x \qquad \\
\boxed{-2x^2 + 2x} \qquad \\
\hline
-0 + 6x - 6 \\
\boxed{-6x + 6} \\
\hline
0
\end{array}
$$

1. x goes into x^3? x^2 times.
2. Multiply (x-1) by x^2.
3. Change sign, Add.
4. Bring down 4x.
5. x goes into $2x^2$? 2x times.
6. Multiply (x-1) by 2x.
7. Change sign, Add
8. Bring down -6.
9. x goes into 6x? 6 times.
10. Multiply (x-1) by 6.
11. Change sign, Add .

Divide.

$$\frac{x^3 - 27}{x - 3}$$

$$x - 3 \overline{)\, x^3 - 27}$$

$$\begin{array}{r} x^2 + 3x + 9 \\ x - 3 \overline{)\, x^3 + 0x^2 + 0x - 27} \\ -x^3 + 3x^2 \qquad\qquad\quad \\ \hline 3x^2 + 0x \qquad\quad \\ -3x^2 + 9x \qquad\quad \\ \hline 9x - 27 \\ -9x + 27 \\ \hline 0 \end{array}$$

Long Division.

$$\begin{array}{r} x+2 \\ x-4\overline{)\,x^2-2x-8} \\ -x^2+4x \\ \hline 2x-8 \\ -2x+8 \\ \hline 0 \end{array}$$

Check

$$(x+2)(x-4)$$

$$= x^2-4x+2x-8$$

$$= x^2-2x-8$$

- How to
- How to use the Remainder Theorem and the Factor Theorem

Synthetic Division

is a shorthand, or shortcut, method of polynomial division in the special case of dividing by a degree one polynomial -- and it only works in this case.

Divide $x^4 - 10x^2 - 2x + 4$ by $x + 3$

-3	1	0	-10	-2	4
		-3	+9	3	-3
	1	-3	-1	1	1

$$\frac{x^4 - 10x^2 - 2x + 4}{x + 3} = x^3 - 3x^2 - x + 1 + \frac{1}{x+3}$$

SYNTHETIC DIVISION: $(5x^3 - 13x^2 + 10x - 8) \div (x - 2)$

STEP #1: Write the Polynomial in DESCENDING ORDER by degree and write any ZERO coefficients for missing degree terms in order

$$\textit{Polynomial Descending Order}: 5x^3 - 13x^2 + 10x - 8$$

STEP #2: Solve the Binomial Divisor = Zero

$$x - 2 = 0; x = 2$$

STEP #3: Write the ZERO-value, then all the COEFFICIENTS of Polynomial.

Coefficients 5 -13 10 -8 = | **Zero = 2**

STEP #4 (Repeat):
(1) ADD Down, (2) MULTIPLY, (3) Product → Next Column

SYNTHETIC DIVISION:

Coefficients	5	-13	10	-8 =	2
	↓	10	-6	8	
	5	-3	4	0 = Remainder	

STEP #5: Last Answer is your REMAINDER

STEP #6: POLYNOMIAL DIVISION QUOTIENT

Write the coefficient "answers" in descending order starting with a Degree ONE LESS THAN Original Degree and include NONZERO REMAINDER OVER DIVISOR at end

(If zero is fraction, then divide coefficients by denominator)

5 -3 4 → $5x^2 - 3x + 4$

$$(5x^3 - 13x^2 + 10x - 8) \div (x - 2) = \boxed{5x^2 - 3x + 4}$$

SYNTHETIC DIVISION: Practice

[1] $(3x^5 - 7x^4 - 4x^2 - 2x - 6)(x-3)^{-1}$

3	–7	0	–4	–2	–6	3
	9	6	18	42	120	
3	2	6	14	40	114	

$$3x^4 + 2x^3 + 6x^2 + 14x + 40 + \frac{114}{x-3}$$

[2] $(8x^4 - 4x^2 + x + 4) \div (2x+1)$

−0.5	8	0	−4	1	4
		−4	2	1	−1
	8	−4	−2	2	3

Divide by 2

$$4x^3 - 2x^2 - x + 1 + \frac{3}{2x+1}$$

[3] $(x^4 - 5x^3 - 13x^2 + 10) \div (x+1)$

[4] $(x^3 + 2x^2 - 5x + 12) \div (x+4)$

REMAINDER THEOREM

The remainder theorem says that if we divide a polynomial $f(x)$ by $x-a$, the remainder is given by $f(a)$

Proof of the Remainder theorem

Let $f(x)$ be a polynomial that is divided by $x - a$

The quotient is another polynomial and the remainder is a constant.

We can write $$\frac{f(x)}{x-a} \equiv g(x) + \frac{R}{x-a}$$

Multiplying by $x-a$ gives

$$f(x) \equiv (x-a)g(x) + R$$

So, $$f(a) = (a-a)g(a) + R$$
$$= R$$

:

Given a polynomial function f(*x*):

then f(a) equals the remainder of $\dfrac{f(x)}{(x-a)}$

Example: Find the given value

[A] $f(x) = x^3 + 3x^2 - 4x - 7$, find $f(2)$

Method #1: Synthetic Division

1	3	- 4	-7	2
↓	2	10	12	
1	5	6	5	

Method #2: Substitution/ Evaluate

$f(2) = (2)^3 + 3(2)^2 - 4(2) - 7$

$f(2) = 8 + 12 - 8 - 7$

$f(2) = 5$

[B] $f(x) = x^4 - 5x^2 + 8x - 3$, find $f(-3)$

1	0	- 5	8	-3	-3
	-3	9	-12	12	
1	-3	4	-4	9	

$f(-3) = (-3)^4 - 5(-3)^2 + 8(-3) - 3$

$f(-3) = 81 - 45 - 24 - 3 = 9$

Ex:Find the remainder when x^3+3x^2-4x+1 is divided by $x-2$

Solution: Let $f(x)=x^3+3x^2-4x+1$

So, $a=2 \Rightarrow R=f(2)$

$$f(2)=(2)^3+3(2)^2-4(2)+1$$

$$=8+12-8+1$$

$$\Rightarrow R=13$$

The Factor Theorem:

When f(a)=0 then x−a is a factor of f(x)

or

When x−a is a factor of f(x) then f(a)=0

<u>FACTOR THEOREM</u>:

$(x - \text{a})$ is a factor of $\text{f}(x)$ iff $\text{f}(a) = 0$

remainder = 0

Example: Factor a Polynomial with Factor Theorem

Given a polynomial and one of its factors, find the remaining factors using synthetic division.

$$\textit{Polynomial}: x^3 + 3x^2 - 36x - 108; \quad \text{Factor} = (x+3)$$

1	3	-36	-108	-3
	-3	0	108	
1	0	-36	0	

$$= x^2 - 36$$

(Synthetic Division)

(x + 6) (x - 6) Remaining factors

$$\textit{Therefore } x^3 + 3x^2 - 36x - 108 = (x+3)(x+6)(x-6)$$

examples:

Given a polynomial and one of its factors, find the remaining factors.

[A] $x^3 + 4x^2 - 15x - 18;$ **Factor** $= (x - 3)$

	1	4	−15	−18	3
		3	21	18	
	1	7	6	0	

$x^2 + 7x + 6$

$(x+6)(x+1)$

STOP once you have a quadratic!

$$(x-3)(x+6)(x+1)$$

[B] $2x^3 + 17x^2 + 23x - 42;$ **Factor** $= (2x + 7)$

−3.5	2	17	23	−42	−3.5
		−7	−35	42	
	2	10	−12	0	

$x^2 + 5x - 6$

$(x+6)(x-1)$

STOP once you have a quadratic!

$$(2x+7)(x+6)(x+1)$$

Example 1: Find ZEROS/ROOTS of a Polynomial by FACTORING: **(1)** Factor by Grouping **(2)** U-Substitution **(3)** Difference of Squares, Difference of Cubes, Sum of Cubes

[A] $f(x) = x^3 + 2x^2 + 4x + 8$

Factor by Grouping

$$= x^2(x+2) + 4(x+2)$$

$$0 = (x^2+4)(x+2)$$

$$x = \{\pm\ 2i,\ -2\}$$

[B] $f(x) = x^3 - 3x^2 + 9x - 27$

Factor by Grouping

$$= x^2(x-3) + 9(x-3)$$

$$0 = (x^2+9)(x-3)$$

$$x = \{\pm\ 3i,\ 3\}$$

[C] $f(x) = x^4 - 16$

$$= (x^2+4)(x^2-4)$$

$$= (x^2+4)(x+2)(x-2)$$

$$\{\pm 2i,\ \pm 2\}$$

[D] $f(x) = x^3 - 27$

$$= (x-3)(x^2+3x+9)$$

$$\left\{3,\ \frac{-3 \pm 3i\sqrt{3}}{2}\right\}$$

Some facts :-

1-If r is a zero of P(x) then $x - r$ will be a factor of p(x).
2-If $x - r$ is a factor of P(x) then r will be a zero of P(x).

3-If P(x) is a polynomial of degree n and r is a zero of P(x) then P(x) can be written in the following form.
$P(x) = (x - r)q(x),$ where q(x) is a polynomial with degree n−1. q(x) can be found by dividing $p(x) by\, x - r$

Roots & Zeros of Polynomials I

How the roots, solutions, zeros, *x*-intercepts and factors of a polynomial function are related.

B. Zero Product Property

- **For all numbers *a* and *b*, if *ab* = 0, then**
- ***a* = 0, *b* = 0, or both *a* and *b* equal 0**

C. Solving a Polynomial Equation

Rearrange the terms to have zero on one side:

$$x^2 + 2x = 15 \quad \Rightarrow \quad x^2 + 2x - 15 = 0$$

Factor:

$$(x+5)(x-3) = 0$$

Set each factor equal to zero and solve:

$$(x+5) = 0 \quad \text{and} \quad (x-3) = 0$$

$$x = -5 \qquad\qquad x = 3$$

The only way that $x^2 + 2x - 15$ can $= 0$ is if $x = -5$ or $x = 3$

D. Factors, Roots, Zeros

For our *Polynomial Function*:

$$y = x^2 + 2x - 15$$

The ***Factors*** are: $(x + 5)$ & $(x - 3)$

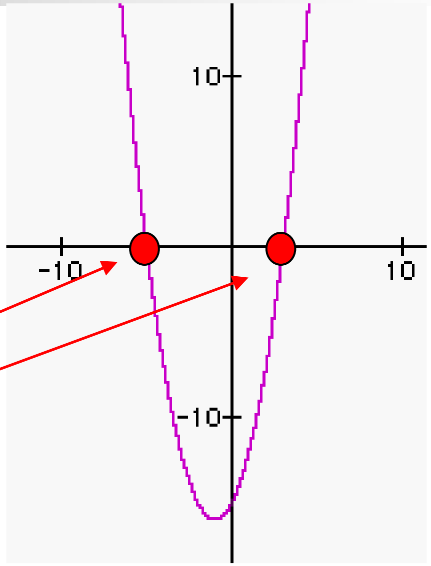
The ***Roots/Solutions*** are: $x = -5$ and 3

The ***Zeros*** are at: $(-5, 0)$ and $(3, 0)$

E. Graph of a Polynomial Function

Here is the graph of our polynomial function:

$$y = x^2 + 2x - 15$$



The *Zeros* of the Polynomial are the values of x when the polynomial equals zero. In other words, the *Zeros* are the x-values where *y equals zero*.

These are also the roots and the x-intercepts.

II. Finding Roots
A. Fundamental Theorem of Algebra

Every Polynomial Equation with a degree higher than zero has at least one root in the set of Complex Numbers.

Note: If P(x)P(x) is a polynomial of degree *n* then P(x) will have exactly nn zeroes, some of which may repeat.

Linear Factorization Theorem

If $f(x)$ is a polynomial of degree n, where $n > 0$, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2) \cdot \cdot \cdot (x - c_n)$$

where $c_1, c_2, \ldots, c_n$ are complex numbers.

Example 1 Zeros of Polynomial Functions

a. The first-degree polynomial $f(x) = x - 2$ has exactly *one* zero: $x = 2$.

b. Counting multiplicity, the second-degree polynomial function

$$f(x) = x^2 - 6x + 9 = (x - 3)(x - 3)$$

has exactly *two* zeros: $x = 3$ and $x = 3$. (This is called a *repeated zero*.)

c. The third-degree polynomial function

$$f(x) = x^3 + 4x = x(x^2 + 4) = x(x - 2i)(x + 2i)$$

has exactly *three* zeros: $x = 0$, $x = 2i$, and $x = -2i$.

d. The fourth-degree polynomial function

$$f(x) = x^4 - 1 = (x - 1)(x + 1)(x - i)(x + i)$$

has exactly *four* zeros: $x = 1$, $x = -1$, $x = i$, and $x = -i$.

Finding EXACT ZEROS (ROOTS) of a Polynomial

[1] **FACTOR** when possible & Identify zeros:

Set each Factor Equal to Zero

[2a] **All Rational Zeros** $= \pm \dfrac{\textit{Factors of } P}{\textit{Factors of } Q}$

P = leading coefficient, Q = Constant of polynomial

[2b] Use **SYNTHETIC DIVISION**

(repeat until you have a quadratic)

[3] **Identify the remaining zeros**

→ Solve the quadratic = 0

(1) factor (2) quad formula (3) complete the square

Answers must be exact, so factoring and graphing won't always work!

The Rational Zero Test

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdot \cdot \cdot + a_2 x^2 + a_1 x + a_0$ has *integer* coefficients, every rational zero of f has the form

$$\text{Rational zero} = \frac{p}{q}$$

where p and q have no common factors other than 1, and

p = a factor of the constant term a_0

q = a factor of the leading coefficient a_n.

$$\text{Possible rational zeros} = \frac{\text{factors of constant term}}{\text{factors of leading coefficient}}$$

Example 2: Find ZEROS/ROOTS of a Polynomial by SYNTHETIC DIVISION (Non-Calculator)

- Find all values of $\frac{P}{Q}$
- Check each value by synthetic division

[A] $f(x) = x^3 - 3x - 2$

Possible Zeros (P/Q)

$\pm 1, \pm 2$

$\mathbf{1 \quad 0 \quad -3 \quad -2}$

[B] $f(x) = x^3 + 3x^2 - 25x + 21$

Possible Zeros (P/Q)

$\pm 1, \pm 3, \pm 7, \pm 21$

$\mathbf{1 \quad 3 \quad -25 \quad 21}$

Example 2: PRACTICE

[C] $f(x) = x^4 + 10x^3 + 33x^2 + 38x + 8$

Possible Zeros (P/Q)

$\pm 1, \pm 2, \pm 4, \pm 8$

$$1 \quad 10 \quad 33 \quad 38 \quad -8$$

[D] $f(x) = x^3 - 3x^2 + x - 3$

Possible Zeros (P/Q)

$\pm 1, \pm 3$

$$1 \quad -3 \quad 9 \quad -27$$

Example 2: PRACTICE

[E] $f(x) = 2x^3 + 3x^2 - 4x - 4$

Possible Zeros (P/Q)

$\pm 1,\ \pm 2,\ \pm 4,\ \pm {}^{1}/_{2}$

$$\begin{array}{cccc|c} 2 & 3 & -4 & -4 & \underline{\quad} \end{array}$$

[F] $f(x) = 2x^4 - 7x^3 + 4x^2 + 7x - 6$

Possible Zeros (P/Q)

$\pm 1,\ \pm 2,\ \pm 3,\ \pm 6,\ \pm {}^{1}/_{2},\ \pm {}^{3}/_{2}$

$$\begin{array}{ccccc|c} 2 & -7 & 4 & 7 & -6 & \underline{\quad} \end{array}$$

Example 2: PRACTICE

[G] $f(x) = 6x^3 + 5x^2 - 3x - 2$

Possible Zeros (P/Q)

$\pm 1, \pm 2, \pm {}^1/_2\ \pm {}^1/_3, \pm {}^2/_3, \pm {}^1/_6$

$$6 \quad 5 \quad -3 \quad -2$$

[H] $f(x) = 3x^3 - 4x^2 - 17x + 6$

Possible Zeros (P/Q)

$\pm 1, \pm 2, \pm 3, \pm 6, \pm {}^1/_3, \pm {}^2/_3$

3 –4 –17 6

Find the rational zeros of $f(x) = x^4 - x^3 + x^2 - 3x - 6$.

Solution

Because the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Possible rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6$

By applying synthetic division successively, you can determine that $x = -1$ and $x = 2$ are the only two rational zeros.

$$
\begin{array}{r|rrrrr}
-1 & 1 & -1 & 1 & -3 & -6 \\
 & & -1 & 2 & -3 & 6 \\
\hline
 & 1 & -2 & 3 & -6 & 0
\end{array}
$$

→ 0 remainder, so $x = -1$ is a zero.

$$
\begin{array}{r|rrrr}
2 & 1 & -2 & 3 & -6 \\
 & & 2 & 0 & 6 \\
\hline
 & 1 & 0 & 3 & 0
\end{array}
$$

→ 0 remainder, so $x = 2$ is a zero.

So, $f(x)$ factors as

$$f(x) = (x + 1)(x - 2)(x^2 + 3).$$

Because the factor $(x^2 + 3)$ produces no real zeros, $x = -1$ and $x = 2$ are the only *real* zeros of f, Find the rational zeros of $f(x) = 2x^3 + 3x^2 - 8x + 3$.

Solution

The leading coefficient is 2 and the constant term is 3.

$$\textit{Possible rational zeros: } \frac{\text{Factors of } 3}{\text{Factors of } 2} = \frac{\pm 1, \pm 3}{\pm 1, \pm 2} = \pm 1, \pm 3, \pm\frac{1}{2}, \pm\frac{3}{2}$$

By synthetic division, you can determine that $x = 1$ is a rational zero.

$$\begin{array}{r|rrrr} 1 & 2 & 3 & -8 & 3 \\ & & 2 & 5 & -3 \\ \hline & 2 & 5 & -3 & 0 \end{array}$$

So, $f(x)$ factors as

$$\begin{aligned} f(x) &= (x - 1)(2x^2 + 5x - 3) \\ &= (x - 1)(2x - 1)(x + 3) \end{aligned}$$

and you can conclude that the rational zeros of f are $x = 1$, $x = \frac{1}{2}$, and $x = -3$.

Complex Zeros Occur in Conjugate Pairs

Let $f(x)$ be a polynomial function that has *real coefficients*. If $a + bi$, where $b \neq 0$, is a zero of the function, the conjugate $a - bi$ is also a zero of the function.

Find a fourth-degree polynomial function with real coefficients that has -1, -1, and $3i$ as zeros.

Solution

Because $3i$ is a zero *and* the polynomial is stated to have real coefficients, you know that the conjugate $-3i$ must also be a zero. So, from the Linear Factorization Theorem, $f(x)$ can be written as

$$f(x) = a(x + 1)(x + 1)(x - 3i)(x + 3i).$$

For simplicity, let $a = 1$ to obtain

$$\begin{aligned} f(x) &= (x^2 + 2x + 1)(x^2 + 9) \\ &= x^4 + 2x^3 + 10x^2 + 18x + 9. \end{aligned}$$

Find all the zeros of $f(x) = x^4 - 3x^3 + 6x^2 + 2x - 60$ given that $1 + 3i$ is a zero of f.

Because complex zeros occur in conjugate pairs, you know that $1 - 3i$ is also a zero of f. This means that both

$$[x - (1 + 3i)] \quad \text{and} \quad [x - (1 - 3i)]$$

are factors of f. Multiplying these two factors produces

$$\begin{aligned}[x - (1 + 3i)][x - (1 - 3i)] &= [(x - 1) - 3i][(x - 1) + 3i] \\ &= (x - 1)^2 - 9i^2 \\ &= x^2 - 2x + 10.\end{aligned}$$

Using long division, you can divide $x^2 - 2x + 10$ into f to obtain the following.

$$\begin{array}{r}
x^2 - x - 6 \\
x^2 - 2x + 10 \overline{\smash{)}\, x^4 - 3x^3 + 6x^2 + 2x - 60} \\
\underline{x^4 - 2x^3 + 10x^2 } \\
-x^3 - 4x^2 + 2x \\
\underline{-x^3 + 2x^2 - 10x } \\
-6x^2 + 12x - 60 \\
\underline{-6x^2 + 12x - 60} \\
0
\end{array}$$

So, you have

$$\begin{aligned}f(x) &= (x^2 - 2x + 10)(x^2 - x - 6) \\ &= (x^2 - 2x + 10)(x - 3)(x + 2)\end{aligned}$$

and you can conclude that the zeros of f are $x = 1 + 3i$, $x = 1 - 3i$, $x = 3$, and $x = -2$.

Write $f(x) = x^5 + x^3 + 2x^2 - 12x + 8$ as the product of linear factors, and list all of its zeros.

Solution

The possible rational zeros are ± 1, ± 2, ± 4, and ± 8. Synthetic division produces the following.

$$\begin{array}{r|rrrrrr} 1 & 1 & 0 & 1 & 2 & -12 & 8 \\ & & 1 & 1 & 2 & 4 & -8 \\ \hline & 1 & 1 & 2 & 4 & -8 & 0 \end{array} \longrightarrow \text{1 is a zero.}$$

$$\begin{array}{r|rrrrr} -2 & 1 & 1 & 2 & 4 & -8 \\ & & -2 & 2 & -8 & 8 \\ \hline & 1 & -1 & 4 & -4 & 0 \end{array} \longrightarrow -2 \text{ is a zero.}$$

So, you have

$$\begin{aligned} f(x) &= x^5 + x^3 + 2x^2 - 12x + 8 \\ &= (x - 1)(x + 2)(x^3 - x^2 + 4x - 4). \end{aligned}$$

You can factor $x^3 - x^2 + 4x - 4$ as $(x - 1)(x^2 + 4)$, and by factoring $x^2 + 4$ as

$$\begin{aligned} x^2 - (-4) &= \left(x - \sqrt{-4}\right)\left(x + \sqrt{-4}\right) \\ &= (x - 2i)(x + 2i) \end{aligned}$$

Upper and Lower Bound Rules

Let $f(x)$ be a polynomial with real coefficients and a positive leading coefficient. Suppose $f(x)$ is divided by $x - c$, using synthetic division.

1. If $c > 0$ and each number in the last row is either positive or zero, c is an **upper bound** for the real zeros of f.
2. If $c < 0$ and the numbers in the last row are alternately positive and negative (zero entries count as positive or negative), c is a **lower bound** for the real zeros of f.

a is a lower bound for the real zeros of f, and b is an upper bound for them $\Leftrightarrow$ **All** the real zeros of f lie in the interval $[a, b]$.

Show that all the real zeros of $f(x) = 4x^3 - 5x^2 - 7x + 2$ must lie in the interval $[-1, 3]$.

Solution

Use Synthetic Division to divide $f(x)$ by $x - 3$:

3	4	-5	-7	2
		12	21	42
	4	7	14	44

Because $\mathbf{3 > 0}$, and all the entries in the last row are **nonnegative**, 3 an **upper bound** for the real zeros of f.

Use Synthetic Division to divide $f(x)$ by $x-(-1)$:

-1	4	-5	-7	2
		-4	9	-2
	4	-9	2	0

Because $\mathbf{-1 < 0}$, and the entries in the last row **alternate between nonnegative and nonpositive entries**, -1 is a **lower bound** for the real zeros of f.