Lecture: Pascal’s Triangle and the Binomial Theorem

Pascal’s Triangle

The numbers in Pascal’s Triangle are so arranged that they reflect as a triangle. Firstly, 1 is placed at the top, then 1s at both sides of the triangle and then we start putting the numbers in a triangular pattern. The numbers in middle we get in each step by addition of the above two numbers. It is similar to the concept of triangular numbers .

Pascal's triangle is used widely in probability theory, combinatorics, and algebra.

Blaise Pascal was born at Clermont-Ferrand, in the Auvergne region of France on June 19, 1623. In 1653 he wrote the article on the Arithmetical Triangle which today is known as the **Pascal’s Triangle**.

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1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Properties of Pascal’s Triangle

- Each number is the sum of the two numbers above it.
- The outside numbers are all 1.
- The triangle is symmetric.
- The first diagonal shows the counting numbers.
- The sums of the rows give the powers of 2.

Pascal's Triangle Patterns

1) Addition of the Rows: One of the interesting properties of the triangle is that the sum of numbers in a row is equal to $\mathbf{2^n}$ where n corresponds to the number of the row:

$1 = 1 = 2^0$

$1 + 1 = 2 = 2^1$

$1 + 2 + 1 = 4 = 2^2$

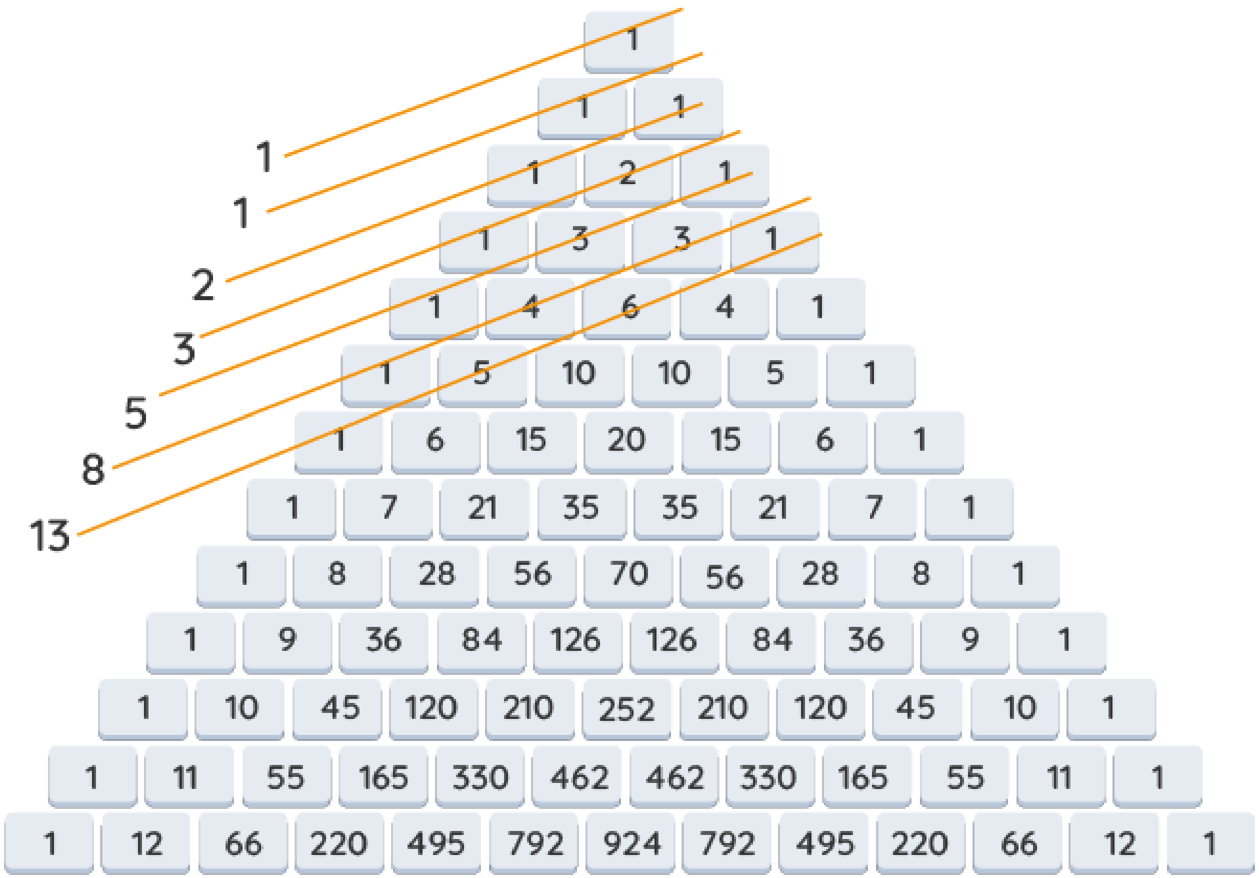
$1 + 3 + 3 + 1 = 8 = 2^3$

$1 + 4 + 6 + 4 + 1 = 16 = 2^4$

2-Fibonacci Sequence in the Triangle:

By adding the numbers in the diagonals of the Pascal triangle the Fibonacci sequence can be obtained as seen in the figure given below.

Fibonacci Sequence in the Pascal's Triangle



The Binomial Theorem

An expression consisting of two terms, connected by + or – sign is called a binomial expression. For example,

$x + a$, $2x - 3y$, $\frac{1}{x} - \frac{1}{y}$, etc., are all binomial

Binomial Coefficients

We know that a *binomial* is a polynomial that has two terms., you will study a formula that provides a quick method of raising a binomial to a power.

look at the expansion of $(x + y)^n$, for several values of n

$$(x+y)^0 = 1,$$

$$(x+y)^1 = x + y,$$

$$(x+y)^2 = x^2 + 2xy + y^2,$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$

$$(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5,$$

$$(x+y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

$$(x+y)^7 = x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + y^7,$$

$$(x+y)^8 = x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8.$$

Binomial Coefficients

1. **In each expansion there are n+1 terms**

2-The sum of the powers of each term is *n*. For instance, in the expansion of

$(x + y)^5$

the sum of the powers of each term is 5.

$4 + 1 = 5$ $3 + 2 = 5$

$$(x + y)^5 = x^5 + \underbrace{5x^4y^1}_{4+1=5} + \underbrace{10x^3y^2}_{3+2=5} + 10x^2y^3 + 5x^1y^4 + y^5$$

2. The coefficients increase and then decrease in a symmetric pattern.

The coefficients of a binomial expansion are called **binomial coefficients.**

Binomial Coefficients

When n is a positive whole number

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \ldots + b^n$$

The Binomial Theorem

In the expansion of $(x+y)^n$

$$(x+y)^n = x^n + nx^{n-1}y + \cdot\cdot\cdot + {}_nC_r x^{n-r}y^r + \cdot\cdot\cdot + nxy^{n-1} + y^n$$

the coefficient of $x^{n-r}y^r$ is

$${}_nC_r = \frac{n!}{(n-r)!r!}.$$

The symbol

$$\binom{n}{r}$$

is often used in place of ${}_nC_r$ to denote binomial coefficients.

Example

Find each binomial coefficient.

a. ${}_8C_2$

b. $\binom{10}{3}$

c. ${}_7C_0$

Solution:

$$\textbf{a. } {}_8C_2 = \frac{8!}{6! \cdot 2!} = \frac{(8 \cdot 7) \cdot \cancel{6!}}{\cancel{6!} \cdot 2!} = \frac{8 \cdot 7}{2 \cdot 1} = 28$$

$$\textbf{b. } \binom{10}{3} = \frac{10!}{7! \cdot 3!} = \frac{(10 \cdot 9 \cdot 8) \cdot \cancel{7!}}{\cancel{7!} \cdot 3!} = \frac{10 \cdot 9 \cdot 8}{3 \cdot 2 \cdot 1} = 120$$

$$\textbf{c. } {}_7C_0 = \frac{\cancel{7!}}{\cancel{7!} \cdot 0!} = 1$$

Example

Write the expansion of the expression $(x + 1)^3$.

Solution:

The binomial coefficients are

$${}_3C_0 = 1,\ {}_3C_1 = 3,\ {}_3C_2 = 3, \text{ and } {}_3C_3 = 1.$$

Therefore, the expansion is as follows.

$$(x + 1)^3 = (1)x^3 + (3)x^2(1) + (3)x(1^2) + (1)(1^3)$$

$$= x^3 + 3x^2 + 3x + 1$$

Binomial Expansions

Sometimes you will need to find a specific term in a binomial expansion.

Instead of writing out the entire expansion, you can use the fact that, from the Binomial Theorem $(x + y)^n$, the $(r + 1)$th term is

$${}_nC_r\, x^{n-r}y^r.$$

Example – *Finding a Term or Coefficient in a Binomial Expansion*

a. Find the sixth term of $(a + 2b)^8$.

Solution:

a. Because the formula is for the $(r + 1)$th term, r is one less than the number of the term you need. So, to find the sixth term in this binomial expansion, use $r = 5$, $n = 8$, $x = a$ and $y = 2b$.

$$ {}_nC_r\, x^{n-r}y^r = {}_8C_5 a^{8-5}(2b)^5 $$

$$ = 56 \cdot a^3 \cdot (2b)^5 $$

$$ = 56(2^5)a^3b^5 $$

$$ = 1792a^3b^5 $$

Example

find the first three terms in the expansion of $(3 - 5z)^{14}$.

Pascal's Triangle

We will now see how useful the triangle can be when we want to expand a binomial expression.

Pascal's Triangle and Coefficients of Binomial

Using Pascal's triangle to expand a binomial expression

Pascal noticed that the numbers in this triangle are precisely the same numbers as the coefficients of binomial expansions, as follows.

$$(x+y)^0 = 1 \qquad \text{0th row}$$

$$(x+y)^1 = 1x + 1y \qquad \text{1st row}$$

$$(x+y)^2 = 1x^2 + 2xy + 1y^2 \qquad \text{2nd row}$$

$$(x+y)^3 = 1x^3 + 3x^2y + 3xy^2 + 1y^3 \qquad \text{3rd row}$$

$$(x+y)^4 = 1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4 \qquad \vdots$$

$$(x+y)^5 = 1x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + 1y^5$$

$$(x+y)^6 = 1x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + 1y^6$$

$$(x+y)^7 = 1x^7 + 7x^6y + 21x^5y^2 + 35x^4y^3 + 35x^3y^4 + 21x^2y^5 + 7xy^6 + 1y^7$$

Example : Use Pascal's Triangle to expand $(2a + b)^4$.

1	0^{th} row
1 1	1^{st} row
1 2 1	2^{nd} row
1 3 3 1	3^{rd} row
1 4 6 4 1	4^{th} row

$$(2a + b)^4 = 1(2a)^4 + 4(2a)^3b + 6(2a)^2b^2 + 4(2a)b^3 + 1b^4$$

$$= 1(16a^4) + 4(8a^3)b + 6(4a^2b^2) + 4(2a)b^3 + b^4$$

$$= 16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4$$

Exercises

1. Use the binomial theorem to expand (a) $(1+x)^4$ and (b) $(1+x)^5$.
2. Use the binomial theorem to expand $(1+2x)^3$.
3. Use the binomial theorem to expand $(1-3x)^4$.
4. Use the binomial theorem to find the first three terms in ascending powers of x of $(1-\dfrac{x}{2})^8$.
5. Find the coefficient of x^5 in the expansion of $(1+4x)^9$.
6. In the expansion of $(1-x)^8$ find the coefficient of x^7.
7. Find the first four terms in the expansion of $\left(2+\frac{x}{3}\right)^{12}$.