Permutations and Combinations

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Permutations and Combinations

The study of permutations and combinations is concerned with determining the number of different ways of arranging and selecting objects out of a given number of objects, without actually listing them.

1-Permutations

A permutation is a mathematical technique that determines the number of possible arrangements in a set when the order of the arrangements matters

OR

A permutation is an ordered arrangement of the elements of some set S

Example :-Let S = {a, b, c}

c, b, a is a permutation of S
b, c, a is a different permutation of S

- An *r*-permutation is an ordered arrangement of *r* elements of the set

The notation for the number of r-permutations: P(n,r) or P_r^n

$$P_r^n = P(n,r) = n(n-1)(n-2)\ldots(n-r+1)$$

$$= \frac{n!}{(n-r)!}$$

$$= \prod_{i=n-r+1}^{n} i$$

There are :-
n ways to choose the first element
n-1 ways to choose the second
n-2 ways to choose the third
…
n-r+1 ways to choose the r^{th} element

By the product rule, that gives us:
P(n,r) = n(n-1)(n-2)…(n-r+1)

Where:

- **n – the total number of elements in a set**
- **r – the number of elements arranged in a specific order**
- **! – factorial**

Type of Permutations

1-Permutation of n different objects (when repetition is not allowed)

In the case of permutation without repetition, the number of available choices will be reduced each time. It can also be represented as:

When all the elements are distinct and few of them are arranged

$$P_r^n = \frac{n!}{(n-r)!}$$

When all the elements are distinct and all of them are arranged

Example:-How many different ways can you arrange 8 students in 5 desks ?

2-Repetition, where repetition is allowed.

. The permutation with repetition of objects can be written using the exponent form.When the number of object is **“n,”** and we have **“r”** to be the selection of object, then;

Choosing an object can be in n different ways (each time).

Thus, the permutation of objects when repetition is allowed will be equal to,

$n \times n \times n \times \dots\dots (r \text{ times}) = n^r$

Example: in the lock, there are 10 numbers to choose from (0,1,2,3,4,5,6,7,8,9) and we choose 3 of them:

$10 \times 10 \times \dots (3 \text{ times}) = 10^3 = 1{,}000$ permutations

3-Permutation when the objects are not distinct (Permutation of multi sets)

The number of distinct permutations of *n things, of which* n_1 are of one kind, n_2 are of a second kind,..., n_k are of kth kind,is

$$\frac{n!}{n_1!\, n_2! \cdots n_k!}$$

<u>Example</u>. How many distinct arrangements exist for the word `hall`?

$\frac{4!}{2!}$ = 12

Example……

- Find the permutations of the word ***Mississippi.***
- Number of Letters
  - 11 – Total Letters
  - 1 – M
  - 4 – I
  - 4 – S
  - 2 - P
- Answer:

$$\frac{11!}{(1!4!4!2!)} = 34650$$

- You can eliminate the 1!'s because they are equal to 1.

Combinations

Let us now assume that there is a group of 3 lawn tennis players X, Y, Z. A team consisting of 2 players is to be formed. In how many ways can we do so? Is the team of X and Y different from the team of Y and X ? Here, order is not important. In fact, there are only 3 possible ways in which the team could be constructed.

- The number of r-combinations of a set with n elements , denoted by C_r^n or ${}^nC_r$, where n is non-negative and $0 \le r \le n$ is:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

In combinations the order does not matter (important)

Corresponding to each combination of ${}^nC_r$, we have r! permutations, because r objects in every combination can be rearranged in r! ways. Hence, the total number of permutations of n different things taken r at a timeis ${}^nC_r \times r!$. On the other hand, it is ${}^nP_r$. Thus

Combinations with Repetition

- *Combinations help us to answer the question "In how many ways can we choose r objects from n objects?"*
- *Now, consider the slightly different question: "In how many ways can we choose r objects from n kinds of objects?*
- **These Questions Are Quite Different:**
- For first question, once we pick one of the n objects, we cannot pick the same object again
- For second question, once we pick one of the n kinds of objects, we can pick the same type of object again!

Combination with repetition allows answering the latter type of question!

Relationship between Permutation and Combination:

Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be
AB, AC, AD, BC, BD, CD.
Here, AB and BA are the same combination as order does not alter the combination.
This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time,
i.e.,
${}^{4}\mathrm{C}_{2} = 6$.
Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways.
Hence,
the number of permutations = ${}^{4}\mathrm{C}_{2} \times 2!$.

Relationship between Permutation and Combination: $P_r^n = {}^{\mathrm{n}}\mathrm{C}_{\mathrm{r}} \times r!$

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!/(n-r)!}{r!/(r-r)!} = \frac{n!}{r!(n-r)!}$$

Example :-

There are 12 females and 18 males in a class. The principal wishes to meet with a group of 5 students to discuss graduation.

a) How many selections are possible?

b) How many selections are possible if the group consists of two females and three males?

a) The question involves choosing 5 students out of 30. In this group, the order of selection is unimportant. So, this is a combinations problem. Use the combinations formula.

Substitute $n = 30$ and $r = 5$ into ${}_nC_r = \dfrac{n!}{(n-r)!r!}$:

$$ {}_{30}C_5 = \frac{30!}{(30-5)!5!} $$

$$ = \frac{30!}{25!5!} $$

$$ = \frac{(\overset{1}{\cancel{30}})(29)(\overset{7}{\cancel{28}})(27)(26)(\overset{1}{\cancel{25!}})}{\underset{1}{\cancel{25!}}(\underset{1}{\cancel{5}})(\underset{1}{\cancel{4}})(\underset{1}{\cancel{3}})(\underset{1}{\cancel{2}})(1)} $$

$$ = 142\ 506 $$

b) There are ${}_{12}C_2$ ways of selecting two female students.

There are ${}_{18}C_3$ ways of selecting three male students.

Using the fundamental counting principle, the number of ways of selecting two females and three males is

$$ {}_{12}C_2 \times {}_{18}C_3 = \frac{12!}{(12-2)!2!} \times \frac{18!}{(18-3)!3!} $$

Why are the elements ${}_{12}C_2$ and ${}_{18}C_3$ multiplied together?

$$ = \frac{(\overset{6}{\cancel{12}})(11)(\overset{1}{\cancel{10!}})}{(\underset{1}{\cancel{10!}})(\underset{1}{\cancel{2}})(1)} \times \frac{(\overset{3}{\cancel{18}})(17)(16)(\overset{1}{\cancel{15!}})}{(\underset{1}{\cancel{15!}})(\underset{1}{\cancel{3}})(\underset{1}{\cancel{2}})(1)} $$

$$ = 66 \times 816 $$

$$ = 53\ 856 $$

There are 53 856 ways to select a group consisting of 2 females and 3 males.

a) Express as factorials and simplify $\frac{{}_{n}C_{5}}{{}_{n-1}C_{3}}$.

b) Solve for n if $2\left({}_{n}C_{2}\right) = {}_{n+1}C_{3}$.

c) Verify that ${}_{n}C_{r} = {}_{n}C_{n-r}$.

Solution

a)

$$\frac{{}_{n}C_{5}}{{}_{n-1}C_{3}} = \frac{\dfrac{n!}{(n-5)!5!}}{\dfrac{(n-1)!}{(n-4)!3!}}$$

What is the formula for ${}_{n}C_{r}$?

Why is $(n-4)!$ in the lower denominator?

$$= \left(\frac{n!}{(n-5)!5!}\right)\left(\frac{(n-4)!3!}{(n-1)!}\right)$$

$$= \frac{n\cancel{(n-1)!}^{1}}{\cancel{(n-5)!}_{1}(5)(4)\cancel{(3!)}_{1}} \times \frac{(n-4)\cancel{(n-5)!}^{1}\cancel{3!}^{1}}{\cancel{(n-1)!}_{1}}$$

Explain why $n!$ can be written as $n(n-1)!$.

$$= \frac{n(n-4)}{20}$$

b)

$$2\left({}_{n}C_{2}\right) = {}_{n+1}C_{3}$$

$$\cancel{2}^{1}\left(\frac{n!}{\cancel{(n-2)!}_{1}\cancel{2!}_{1}}\right) = \frac{(n+1)!}{\cancel{(n-2)!}_{1}3!}$$

$$n! = \frac{(n+1)!}{3!}$$

$$3! = \frac{(n+1)!}{n!}$$

$$6 = \frac{(n+1)\cancel{(n!)}^{1}}{\cancel{n!}_{1}}$$

$$6 = n + 1$$

$$5 = n$$

Verify that ${}_nC_r = {}_nC_{n-r}$.

- Let n and r be non-negative integers with $r \leq n$.
- Then $C(n,r) = C(n,n-r)$

- Proof:

$$C(n,r) = \frac{n!}{r!(n-r)!}$$

$$C(n,n-r) = \frac{n!}{(n-r)![n-(n-r)]!}$$

$$= \frac{n!}{r!(n-r)!}$$

Example:-**From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?**

Solution :-

Since at least 3 men must be chosen, we consider all committees which include 3, 4, and 5 men, with 2, 1, and 0 women, respectively. That is, we want to add the number of ways to:

Choose 3 from 7 men and 2 from 6 women
Choose 4 from 7 men and 1 from 6 women
Choose 5 from 7 men and 0 from 6 women

This is given by:

$$\binom{7}{3}\binom{6}{2}+\binom{7}{4}\binom{6}{1}+\binom{7}{5}\binom{6}{0}$$

$=35*15+35*6+21*1$

$=756$

a) Express in factorial notation and simplify $\left({}_{n-1}C_3\right)\left(\frac{1}{{}_{n-2}C_3}\right)$.

b) Solve for n if $720\left({}_nC_5\right) = {}_{n+1}P_5$.

c)Give an algebraic proof for the.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$