The Summation and Product Notations

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Notation

Summation

Product

Factorial

$$\Gamma(t) = \int_0^\infty x^{t-1} e^{-x}\, dx.$$

$$\Gamma(n) = (n-1)!.$$

$$\Gamma(1) = 0!$$

$$(n+1)! = (n+1)(n!$$

Outline lecture

- Definition of $\sum$ and $\prod$
- Properties of $\sum$ and $\prod$
- Examples.
- Exercises .

Summation Notation

- the Greek letter *sigma* ∑ is generally used to denote a sum of multiple terms. : which denotes the word sum. The summation notation was introduced in 1772 by the French mathematician Joseph Louis (1736–1813)

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n$$

n is called the upper limit of the summation

k is called the index of summation

a_k is the formula used to generate each term of the sum

Summation Notation

- Any letter can be used to index a summation
- The index i is a dummy variable; we can use any variable as the index without affecting the value of the sum, so

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j = \sum_{k=1}^{n} a_k$$

Working in reverse -

Write this series in sigma notation

3 + 6 + 11 + 18 + 27

3 + 6 + 11 + 18 + 27

= (1+2) + (4+2) + (9+2) + (16+2) + (25+2)

= $(1^2 +2)+ (2^2+2) + (3^2+2) + (4^2+2) + (5^2+2)$

$$\sum_{r=1}^{r=5}(r^2+2)$$

Example

1. Write down all the terms of the series

(a) $\sum_{r=1}^{r=5} r^2$ $= 1^2 + 2^2 + 3^2 + 4^2 + 5^2$

$= 1 + 4 + 9 + 16 + 25 = 55$

(b) $\sum_{r=1}^{r=6} (3r - 1)$ $= (3\times1-1)+(3\times2-1) +(3\times3-1) +(3\times4-1) +(3\times5-1) + (3\times6-1)$

$= 2 + 5 + 8 + 11 + 14 + 17 = 57$

(c) $\sum_{r=1}^{r=4} (2r^2 + 3)$ $= (2\times1^2+3)+ (2\times2^2+3)+ (2\times3^2+3)+ (2\times4^2+3)$

$= 5+ 11 + 21 + 35 = 72$

Summation Notation

- **Ex:** Evaluate the following sum:

$$\sum_{i=1}^{4}\left(2i+3\right)$$

- **Sol:**

$$\sum_{i=1}^{4}\left(2i+3\right)=\left(2\cdot 1+3\right)+\left(2\cdot 2+3\right)+\left(2\cdot 3+3\right)+\left(2\cdot 4+3\right)$$

$$=5+7+9+11$$

$$=32$$

Summation Notation

- **Ex:** Evaluate the following sum:

$$\sum_{j=3}^{6}(2j+3)$$

- **Sol:**

$$\sum_{j=3}^{6}(2j+3)=(2\cdot 3+3)+(2\cdot 4+3)+(2\cdot 5+3)+(2\cdot 6+3)$$

$$=9+11+13+15$$

$$=48$$

Properties of Summation:-

(i)

$$\sum_{k=1}^{n}\left(a_k \pm b_k\right) = \sum_{k=1}^{n} a_k \pm \sum_{k=1}^{n} b_k$$

(ii)

$$\sum_{k=1}^{n} c \cdot a_k = c\sum_{k=1}^{n} a_k$$

Note:

$$\sum_{k=1}^{n}\left(a_k \cdot b_k\right) \neq \sum_{k=1}^{n}\left(a_k\right) \cdot \sum_{k=1}^{n}\left(b_k\right)$$

iii) $$\sum_{k=1}^{n}(a_k) = \sum_{k=1}^{m}(a_k) + \sum_{k=m+1}^{n}(a_k) \qquad 1 \le m < n$$

(iv) $$\sum_{i=1}^{n} a = an.$$ where c is any constant .

Suppose our list has just 5 numbers, and they are 1,3,2,5,6. Evaluate $\sum_{i=1}^{5} X_i^2$ $\left(\sum_{i=1}^{5} X_i\right)^2$

$$\sum_{i=1}^{5} X_i^2 \quad 1^2 + 3^2 + 2^2 + 5^2 + 6^2 = 75$$

$$\left(\sum_{i=1}^{5} X_i\right)^2 \quad (1+3+2+5+6)^2 = 17^2 = 289$$

$$\left(\sum_{i=1}^{5} X_i\right)^2 \neq \sum_{i=1}^{5} X_i^2$$

Summation Notation

- **<u>Ex:</u>** Evaluate the following sum

- **<u>Sol:</u>**

$$\sum_{k=1}^{6}\left(1.2^{k}-1.2^{k-1}\right)$$

$$\sum_{k=1}^{6}\left(1.2^{k}-1.2^{k-1}\right)=\begin{array}{l}\left(1.2-1.2^{1-1}\right)+\quad\left(1.2^{2}-1.2^{2-1}\right)+\quad\left(1.2^{3}-1.2^{3-1}\right)+\\ \left(1.2^{4}-1.2^{4-1}\right)+\quad\left(1.2^{5}-1.2^{5-1}\right)+\quad\left(1.2^{6}-1.2^{6-1}\right)\end{array}$$

$$=0.2+0.24+0.288+0.3456+0.41472+0.497664$$

$$=1.985984$$

Write the following summation in expanded form: $\sum_{i=0}^{n} \frac{(-1)^i}{i+1}$.

Solution:

$$\sum_{i=0}^{n} \frac{(-1)^i}{i+1} = \frac{(-1)^0}{0+1} + \frac{(-1)^1}{1+1} + \frac{(-1)^2}{2+1} + \frac{(-1)^3}{3+1} + \cdots + \frac{(-1)^n}{n+1}$$

$$= \frac{1}{1} + \frac{(-1)}{2} + \frac{1}{3} + \frac{(-1)}{4} + \cdots + \frac{(-1)^n}{n+1}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^n}{n+1}$$

Product Notation

The notation for the product of a sequence of numbers is analogous to the notation for their sum. The Greek capital letter pi, $\prod$, denotes a product.

For example,

$$\prod_{k=1}^{5} a_k = a_1 a_2 a_3 a_4 a_5.$$

• Definition

If m and n are integers and $m \le n$, the symbol $\prod_{k=m}^{n} a_k$, read the **product from *k* equals *m* to *n* of *a*-sub-*k***, is the product of all the terms $a_m,\ a_{m+1},\ a_{m+2},\ \ldots,\ a_n$. We write

$$\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1} \cdot a_{m+2} \cdots a_n.$$

Product Notation

A recursive definition for the product notation is the following: If *m* is any integer, then

$$\prod_{k=m}^{m} a_k = a_m \qquad \text{and} \qquad \prod_{k=m}^{n} a_k = \left(\prod_{k=m}^{n-1} a_k\right) \cdot a_n \quad \text{for all integers } n > m.$$

Example *Computing Products1) 2)*

$$\prod_{k=1}^{5} k \qquad \prod_{k=1}^{1} \frac{k}{k+1}$$

$$\prod_{k=1}^{5} k = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \quad = 120$$

$$\prod_{k=1}^{1} \frac{k}{k+1} = \frac{1}{1+1} \quad = \frac{1}{2}$$

Factorial Notation

- **Definition**

For each positive integer n, the quantity ***n* factorial** denoted ***n*!,** is defined to be the product of all the integers from 1 to n:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

Zero factorial, denoted $0!$, is defined to be 1:

$$0! = 1.$$

$$n! = \begin{cases} 1 & \text{if } n = 0 \\ n \cdot (n-1)! & \text{if } n \geq 1. \end{cases}$$

Examples:

4! = 4•3 •2 •1 = 24

$$n! = \prod_{i=1}^{n} i$$

a factorial of n items gives the number of ways can arrange the given items.

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120 = \prod_{i=1}^{5} i$$

Example

- Evaluate each factorial expression:

$$\frac{9!}{3!7!} \qquad \frac{3!8!}{4!4!} \qquad \frac{(n+1)!}{n!}$$

- How many different ways can you arrange the letters in the word (cat)

Mathematical Induction

Proof by mathematical induction is very useful method in proving the validity of a mathematical statement $(\forall n, p(n))$involving integers n greater than or equal to some initial integerx_0 .

Principle of mathematical induction

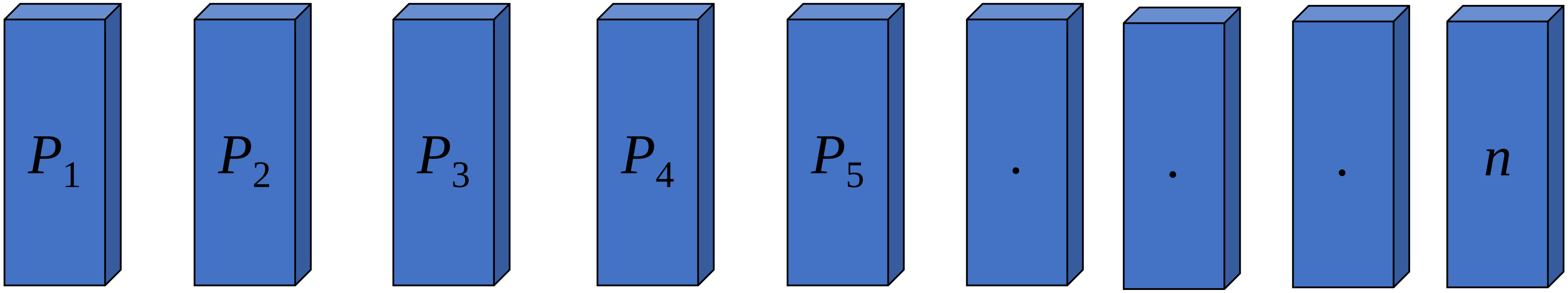
If p(n) is a sentence involving the natural number n such that

1) $P(1)$ is true , and

2) $p(k)$ is true
$\Longrightarrow p(k+1) is\ true\ for\ any\ arbitrary\ natural\ number$
n

That is "$(\forall n)P(n)$"

Then p(n) is true for every natural number n.



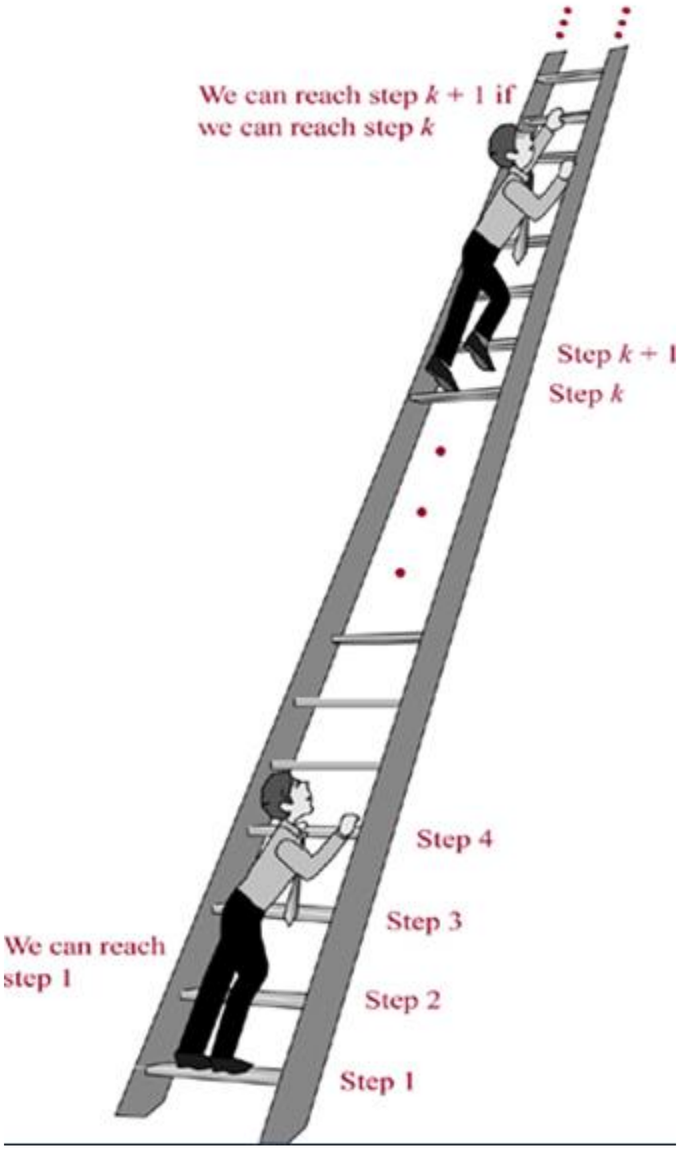
We know that:

using Mathematical Induction which is a rule of inference that tells us:

P(1)

$\forall k\ (P(k) \rightarrow P(k+1))$

$\therefore \forall n\ (P(n)$



Induction

If we have a propositional function P(n), and we want to prove that P(n) is true for any natural number n, we do the following:

Show that P(k_0) is true.
(basis step)

Show that if P(k) then P($k + 1$) for any k∈N.
(inductive step)

Then P(n) must be true for any n∈N.
(conclusion)

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

Proof :- { by mathematical induction }

let p(n) : $2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1 \forall n \in \mathbb{N} \cup \{0\}$

Base step :- we show that the p(n) is true at n =0

left side (L.S)of $p(0) = 1$

Right side (R.S) $of\ p(0) = 2^{0+1} - 1$

$=1$

Since both sides are equal therefore p(n)is true when n = 0.

Inductive step : Assume that p(n) is true ant n=k .

(i.e $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$) for some $k \in \mathbb{Z}^+$) and prove p(n) is true at k+1

which states that $2^0 + 2^1 + \ldots + 2^{k+1} = 2^{k+2} - 1$ ……(*)

Let us begin with the left side (*)

$$2^0 + 2^1 + \ldots + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1} \text{ (by hypothesis step)}$$

$$= 2\ 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

= right sid (*)

Thus, p(n) holds for n = k + 1,

Conclusion: By mathematical induction

The statement P(n) is true for all nonnegative integers

Example 2:-Prove by mathematical induction that
$n^3 - n$ is divisible by 3 for all natuaral numbers n≥ 1