* Trigonometric Substitutions

- Trig substitutions can be effective in trasforming integrals involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$ and $\sqrt{x^2-a^2}$ into integrals we can evaluate directly.

Expression	substitution	identity
① $\sqrt{a^2-x^2}$	$x=a\sin\theta$, $\theta\in[-\frac{\pi}{2},\frac{\pi}{2}]$	$1-\sin^2\theta=\cos^2\theta$
② $\sqrt{a^2+x^2}$	$x=a\tan\theta$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$ or $x=a\sinh\theta$	$1+\tan^2\theta=\sec^2\theta$
③ $\sqrt{x^2-a^2}$	$x=a\sec\theta$, $\theta\in[0,\frac{\pi}{2})$ or $x=a\cosh\theta$	$\sec^2\theta-1=\tan^2\theta$

eg $\int\frac{dx}{\sqrt{4+x^2}}$, let $x=2\tan\theta$, $dx=2\sec^2\theta\,d\theta$, $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$

$$=\int\frac{2\sec^2\theta\,d\theta}{\sqrt{4+4\tan^2\theta}}=\int\frac{2\sec^2\theta\,d\theta}{\sqrt{4(1+\tan^2\theta)}}=\int\frac{2\sec^2\theta\,d\theta}{2\sqrt{\sec^2\theta}}$$

$$=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C$$

$x=2\tan\theta\Rightarrow\tan\theta=\frac{x}{2}$

$4+x^2=4+4\tan^2\theta=4(1+\tan^2\theta)=4\sec^2\theta$

$\sec^2\theta=\frac{4+x^2}{4}\Rightarrow\sec\theta=\frac{\sqrt{4+x^2}}{2}$; $\sec\theta>0$ in $\theta\in(-\frac{\pi}{2},\frac{\pi}{2})$

$$=\ln\left|\frac{\sqrt{4+x^2}}{2}+\frac{x}{2}\right|+C.$$

* $\int\frac{x^2}{\sqrt{9-x^2}}dx$, let $x=3\sin\theta$, $dx=3\cos\theta\,d\theta$,

$9-x^2=9-9\sin^2\theta=9(1-\sin^2\theta)=9\cos^2\theta$

$$=\int\frac{9\sin^2\theta\cdot3\cos\theta\,d\theta}{3\cos\theta}=9\int\sin^2\theta\,d\theta=9\int\frac{1-\cos2\theta}{2}d\theta$$

$$=\frac{9}{2}\left(\theta-\frac{\sin2\theta}{2}\right)+C=\frac{9}{2}(\theta-\sin\theta\cos\theta)+C$$

$$=\frac{9}{2}\left(\sin^{-1}\frac{x}{3}-\frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right)+C$$

$\theta=\sin^{-1}\frac{x}{3}$, $\sin\theta=\frac{x}{3}$, $\cos\theta=\frac{\sqrt{9-x^2}}{3}$