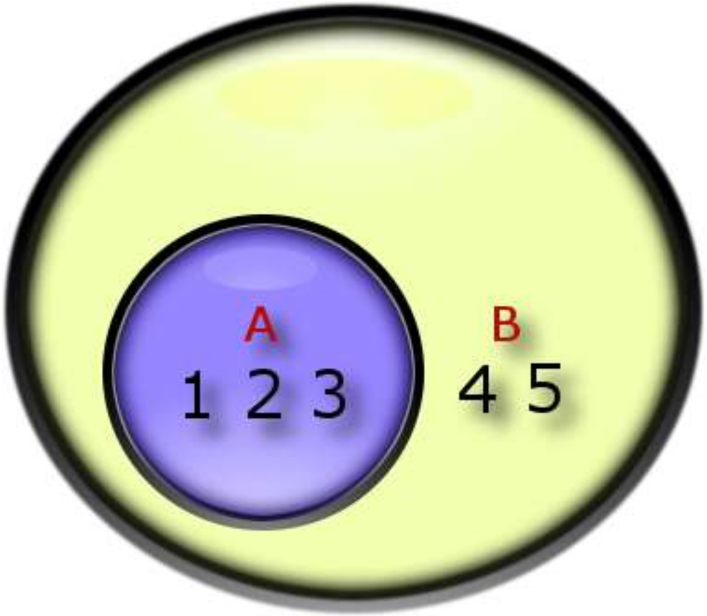
Set Theory



What is a set?

- Sets are used to define the concepts of relations and functions. The study of geometry, sequences, probability, etc. requires the knowledge of sets.

Georg Cantor
(1845-1918)

- The theory of sets was developed by German mathematician ***Georg Cantor*** (1845-1918).
- He first encountered sets while working on "problems on trigonometric series".
- *Studying sets helps us categorize information. It allows us to make sense of a large amount of information by breaking it down into smaller groups.*

Definition: *A **set** is any collection of objects specified in such a way that we can determine whether a given object is or is not in the collection.*

- *In other words* A **set** is a collection of objects.
- These objects are called elements or members of the set.
- The symbol for element is $\in$.
- For example, if you define the set as all the fruit found in my refrigerator, then apple and orange would be elements or members of that set.
- The following points are noted while writing a set.
  - Sets are usually denoted by capital letters A, B, S, etc.
  - The elements of a set are usually denoted by small letters a, b, t, u, etc

Examples:

- A = {a, b, d, 2, 4}
- B = {math, religion, literature, computer science}
- C = $\{\Re, \wp, \Psi, \xi\}$

Sets

- Other ways to denote sets
  - Ellipses
    - **N** = {0, 1, 2, 3, 4. . .}

      (set of **natural numbers)**

    - **Z** = {. . ., -3, -2, -1, 0, 1, 2, 3,. . .}

      (set of **integers)**

    - E = {0, 2, 4, 6. . .}

      (set of **even natural numbers)**

- Sets can be well defined.
- A well defined set is a set whose contents are clearly determined. The set defined as "colors" would not be well defined while "the set of colors in a standard box of eight crayons" is well defined.

There are three methods used to indicate a set:

1. Description
2. Roster form
3. Set-builder notation

- **Description :** Description means just that, words describing what is included in a set.
  - For example, Set M is the set of months that start with the letter J.
- **Roster Form :** Roster form lists all of the elements in the set within braces {element 1, element 2, …}.
  - For example, Set $M = \{$ January, June, July$\}$
- **Set-Builder Notation:** Set-builder notation is frequently used in algebra.
  - For example, $M = \{\, x \mid x \in$ is a month of the year and x starts with the letter J$\}$
  - This is read, "Set M is the set of all the elements x such that x is a month of the year and x starts with the letter J".

Subsets

- A is a **subset** of B if every element of A is also contained in B. This is written

$$A \subset B.$$

For example, the set of integers

$$\{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \}$$

is a subset of the set of real numbers.

Formal Definition:

$A \subset B$ means "if $x \in A$, then $x \in B$."

Empty set

- Set with no elements
- { } or Ø.

Elements may be sets

A = {1,2,{1,3,5},3,{4,6,8}}

B = {{1,2,3},{4,5,6}}

C = {Ø, 1, 3} = {{},1,3}

D = {Ø} = {{}} ≠ Ø

- **Set size**
  - Called ***cardinality***
  - Number of elements in set
  - Denoted $|A|$ is the size of set A
  - If A = {2,3,5,7,8}, then $|A| = 5$
  - If a set A has a finite number of elements, it is a ***finite set***.
  - A set which is not finite is ***infinite***.

Set relations

- $\in$ - "*is a member of*"
  - $x \in A$
- $\subseteq$ - "*subset*"
  - $A \subseteq B$ A is a subset of B
    - Every element in A is also in B
      - $\forall x: x \in A \rightarrow x \in B$

- $\supseteq$ - "*superset*"
  - A $\supseteq$ B
    - A is a superset of B
      - Every element in B is also in A
      - $\forall x: x \in B \rightarrow x \in A$
- $\subset$ - "*proper subset*"
  - A $\subset$ B - A is a proper subset of B (A $\neq$ B)
    - Every element in A is also in B *and*
      - A $\neq$ B
    - $(\forall x: x \in A \rightarrow x \in B) \wedge A \neq B$
- $\supset$ - "*proper superset*"
  - A $\supset$ B - A is a proper superset of B (A $\neq$ B)
    - Every element in B is also in A *and*
      - A $\neq$ B
    - $(\forall x: x \in B \rightarrow x \in A) \wedge A \neq B$
- Example: N $\subseteq$ Z $\subseteq$ Q $\subseteq$ R

Numbers and Set

- There are different types of numbers:
- **Cardinal numbers** - answer the question “How many?”
- ***Ordinal numbers*** - such as first, second, third. . .
- ***Nominal numbers*** – which are used to name things. Examples of nominal numbers would be your driver’s license number or your student ID number.
- *The cardinal number of a set S, symbolized as n(S), is the number of elements in set S.*
- *If S = { blue, red, green, yellow } then n(S) = 4.*
- *Two sets are considered equal sets if they contain exactly the same elements.*
- *Two sets are considered* ***equivalent sets*** *if they contain the same number of elements (if n(A) = n(B)).*

- If $E = \{ 1 , 2 , 3 \}$ and $F = \{ 3 , 2 , 1 \}$, then the sets are equal (since they have the same elements), and equivalent (since they both have 3 elements).

- If $G = \{$ cat , dog , horse , fish $\}$ and $H = \{ 2 , 5 , 7 , 9 \}$, then the sets are not equal (since they do not have the same elements), but they are equivalent (since they both have 4 elements, $n(G) = n(H)$).

Power Sets

- Given any set, we can form a *set of all possible subsets.*
- This set is called the ***power set***.
- Notation: power set or set A denoted as P(A)
- Ex: Let A = {a}
  - P(A) = {Ø, {a}}
- Let A = {a, b}
- P(A) = {Ø, {a}, {b}, { a, b}}

- Let B = {1, 2, 3}

P(B)={Ø,{1},{2},{3},{1,2},{1,3},{2,3},{1,2,3}}

Cartesain Product

- *Ordered pairs* - A list of elements in which the order is significant.
- Order is not significant for sets!
- **Notation:** use *round brackets.*
  - (a, b)
  - $(1, 2)$
  - $(2, 1)$

$$\{a,b\} = \{b,a\}$$
$$(a,b) \neq (b,a)$$

- ***Cartesian Product***: Given two sets A and B, the set of
  – all *ordered pairs* of the form (a , b) where a is any
  – element of A and b any element of B, is called the
  – *Cartesian product* of A and B.
- Denoted as A x B
  - A x B = $\{(a,b) \mid a \in A \text{ and } b \in B\}$
  - Ex: Let A = {1,2,3}; B = $\{x,y\}$
    – A x B = $\{(1,x),(1,y),(2,x),(2,y),(3,x),(3,y)\}$
    – B x A = $\{(x,1),(y,1),(x,2),(y,2),(x,3),(y,3)\}$
    – B x B = $B^2 = \{(x,x),(x,y),(y,x),(y,y)\}$

Set Operators

- ***Union*** of two sets A and B is the set of all elements in <u>either</u> set A or B.
  - Written $A \cup B$.
  - $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

- ***Intersection*** of two sets A and B is the set of all elements in both sets A or B.
  - Written $A \cap B$.
  - $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

- ***Difference*** of two sets A and B is the set of all elements in set A which are not in set B.
  - Written $A - B$.
  - $A - B = \{x \mid x \in A \text{ and } x \notin B\}$
  - also called *relative complement*

- ***Complement*** of a set is the set of all elements not in the set.
  - Written A^c
  - Need a *universe* of elements to draw from.
  - Set **U** is usually called the *universal set.*
  - $A^c = \{x \mid x \in \mathbf{U} - A\}$

- Sets with no common elements are called ***disjoint***
  - If $A \cap B = \varnothing$, then A and B are *disjoint.*

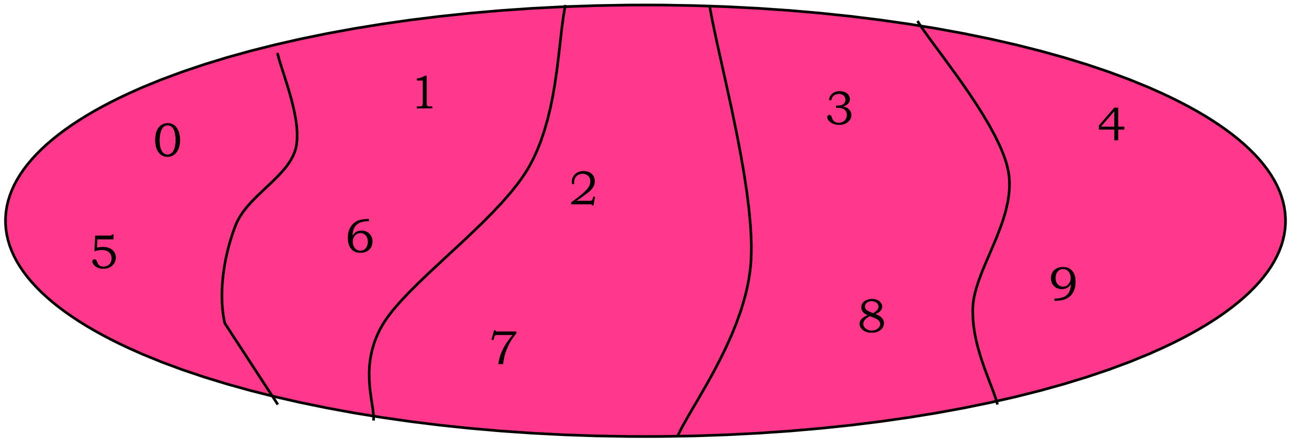
- If $A_1, A_2, \ldots A_n$ are sets, and no two have a common element, then we say they are *mutually disjoint.*
  - $Ai \cup Aj = \varnothing$ for all $i,j \leq n$ and $i \neq j$
  - Consider $M_d = \{x \mid x \in$ MVNC students, $d \in$ MVNC dorm rooms$\}$
  - Consider $M_n = \{x \in I \mid (x \text{ MOD } 5) = n\}$

- ***Partition*** - A collection of disjoint sets which *collectively*
  - Make up a larger set.
  - Ex: Let A = {a,b}; B = {c,d,e}; C = {f,g} and
    - D = {a,b,c,d,e,f,g}
    - Then sets A,B,C form a *partition* of set D

- Let A be a nonempty set ($A \neq \emptyset$), and suppose that $B_1, B_2, B_3, \ldots, B_n$ are *subsets* of A, such that:
  - None of sets $B_1, B_2, B_3, \ldots, B_n$ are empty;
  - The sets $B_1, B_2, B_3, \ldots, B_n$ are *mutually disjoint.* (They have no elements in common)
  - The *union* of sets $B_1, B_2, B_3, \ldots, B_n$ is equal to A.

    e.g. $B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_n = A$

- Then we say the sets B_1, B_2, B_3, . . ., B_n form a ***partition*** of the set A.
- The subsets B_1, B_2, B_3, . . ., B_n are called ***blocks*** of the partition.



Universal Set

- A universal set is the super set of all sets under consideration and is denoted by U.
- Example: If we consider the sets A, B and C as the cricketers of India, Australia and England respectively, then we can say that the universal set (U) of these sets contains all the cricketers of the world.
- The union of two sets A and B is the set which contains all those elements which
- are only in A, only in B and in both A and B, and this set is denoted by "$A \cup B$".
- $A \cup B \{x : x \in A \text{ or } x \in B\}$
- Example: If A = {a, 1, *x*, p} and B = {p, q, 2, *x*},

  then $A \cup B$ = {a, p, q, x, 1, 2}.

  Here, a and 1 are contained only in A; q and 2 are contained only in B; and p and x are contained in both A and B.

Set Properties

- **Property 1** (*Properties of Ø and ∪*)
  - $A \cup \varnothing = A$, $A \cap U = A$
  - $A \cup U = U$, $A \cap \varnothing = \varnothing$
- **Property 2** **(** *The idempotent properties***)**
  - $A \cup A = A$, $A \cap A = A$
- **Property 3** (*The commutative properties*)
  - $A \cup B = B \cup A$, $A \cap B = B \cap A$
- **Property 4** (*The associative properties*)
  - $A \cup (B \cup C) = (A \cup B) \cup C$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- **Property 5** **(***The distributive properties)*
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

- **Property 6** (*Properties of the complement*)
  - $\varnothing^C = U$, $U^C = \varnothing$
  - $A \cup A^C = U$, $A \cap A^C = \varnothing$
  - $(A^C)^C = A$
- **Property 7** (*De Morgan's laws*)
  - $(A \cup B)^C = A^C \cap B^C$
  - $(A \cap B)^C = A^C \cup B^C$
- **Property 8** *(Absortion laws)*
  - $A \cap (A \cup B) = A$
  - $A \cup (A \cap B) = A$