Predicates and Quantifiers

Review

- Proposition:
  1. It is a sentence that declares a fact.
  2. It is either true or false, but not both.

Examples:

- 2 + 1 = 3.

  True Proposition
- Toronto is the capital of Canada.

  False Proposition
- x + 1 = 2.

  Neither true nor false

- Logical Operators
  - Negation

    $\neg p$ "not p."
  - Conjunction

    $p \wedge q$ "p and q."
  - Disjunction

    $p \vee q$ "p or q."
  - Exclusive or

    $p \oplus q$ "p or q, but not both."
  - Conditional statement

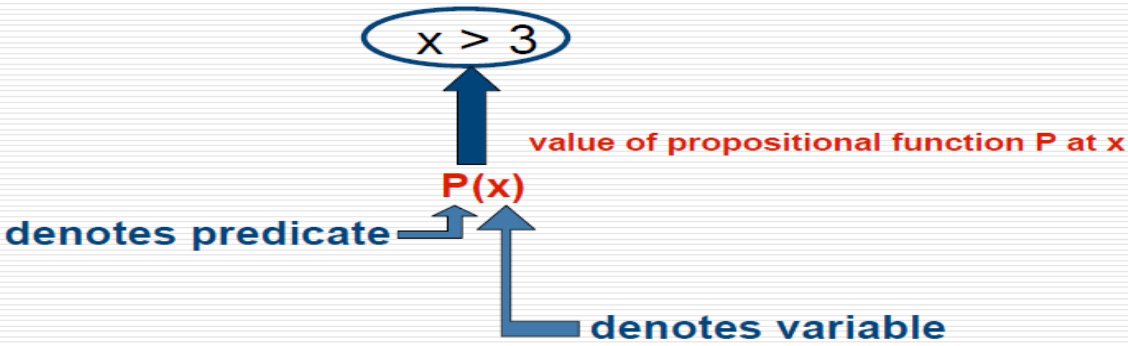
    $p \rightarrow q$ "If p, then q."
  - Biconditional statement

    $p \leftrightarrow q$ "p if and only if q."

Predicate Logic

- More powerful
- Express a wide range of statements in mathematics and computer science

Predicates



Example: x > 3

- The variable x is the subject of the statement
- **Predicate** “is greater than 3” refers to a property that the subject of the statement can have
- Can denote the statement by p(x) where p denotes the predicate “is greater than 3” and x is the variable
- p(x): also called the value of the **propositional function** p at x
- Once a value is *assigned* to the variable x, p(x) becomes a proposition and has a truth value

Predicates (example)

P(x) : x>3.
What are the truth value of P(4) and P(2)?

Solution:

- Set variable x
  - x=4

    P(4): 4>3

    True
  - x=2

    P(2): 2>3

    False

Q(x,y) : x = y+3.
What are the truth value of Q(1,2) and Q(3,0)?

Solution:

- Set variables x and y
  - x=1, y=2

    Q(1,2): 1=2+3

    False
  - x=3, y=0

    Q(3,0): 3=0+3

    True

Quantifiers

- Create a proposition from a propositional function using **Quantifiers**

- Quantifiers express the **range** of elements the statement is about.
  - The universal quantifier
  - The existential quantifier

The universal quantifier

- The universal quantifier is used to assert a **property** of **all** values of a variable in a particular **domain**.

- The universal quantification of $P(x)$ is
  "$P(x)$ for all values of x in the domain.",
  denoted by $\forall x\, P(x)$

  ↑ **The universal quantifier**

- The universal quantifier
  - For all …
  - For every …
  - For each …
  - All of …
  - For arbitrary …

Ex: Let P(x): x+1 > x

The universal quantifier of P(x) is in the domain of real numbers:

$\forall x$ P(x) (x is a real number)

$\forall x$ (x+1 > x) (x is a real number)

- $\forall x$ P(x)
  - When true?
    - P(x) is true for every x in the domain
  - When false?
    - P(x) is not always true when x is in the domain
      (find a value of x that P(x) is false)
- An element for which P(x) is false is called a **counterexample** of $\forall x$ P(x).

P(x): x>3

P(2): 2>3 is a counterexample of $\forall x$ P(x)

$P(x)$: $x+1 > x$.

What is the truth value of $\forall x\, P(x)$ in the domain of real numbers?

Solution:

- Check if $P(x)$ is true for all real numbers
  - "$x+1 > x$" is true for all real number

    So, the truth value of $\forall x\, P(x)$ is true.

$Q(x)$: $x < 2$.

What is the truth value of $\forall x\, Q(x)$ in the domain of real numbers?

Solution:

- Find a counterexample for $\forall x\, Q(x)$
  - $Q(3)$: $3<2$ is false

    $x=3$ is a counterexample for $\forall x\, Q(x)$, so $\forall x\, Q(x)$ is false.

$P(x)$: $x^2 > 0$.

What is the truth value of $\forall x\, P(x)$ in the domain of integers?

Solution:

- Find a counterexample for $\forall x\, P(x)$
  - $P(0)$: $0>0$ is false

    $x=0$ is a counterexample for $\forall x\, P(x)$, so $\forall x\, P(x)$ is false.

$P(x)$: $x^2 \geq x$.

What is the truth value of $\forall x\, P(x)$ in the domain of all real numbers?

Solution: How to find a counterexample?

$x^2 \geq x$.

$(x^2 - x) = x(x - 1) \geq 0$.

x and (x-1) must both be zero or positive.	OR	x and (x-1) must both be zero or negative.
$x \geq 0$ and $(x - 1) \geq 0$		$x \leq 0$ and $(x - 1) \leq 0$
$x \geq 0$ and $x \geq 1$		$x \leq 0$ and $x \leq 1$
$x \geq 1$		**$x \leq 0$**

$0 < x < 1$ such as $x=1/2$ is a counterexample

$P(x)$: $x^2 \geq x$.

What is the truth value of $\forall x\, P(x)$ in the domain of all integers?

Solution:

- Check if $P(x)$ is true for all integers
  - $P(x)$ is true when $x \geq 1$ or $x \leq 0$.
  - There is no integer between $0 < x < 1$.
  - So, $\forall x\, P(x)$ is true for the domain of all integers.

- $\forall x\, P(x)$ in the domain D
- If D can be listed as $x_1, x_2, \ldots, x_n$.

$\forall x\, P(x)$ in the domain D is the same as

$$P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$$

P(x): $x^2 < 10$.

What is the truth value of $\forall x\, P(x)$ in the domain of positive integers not exceeding 4?

Solution:

- List the domain
  - Domain is 1, 2, 3, 4.
- Find the equivalent conjunction and its truth value
  - $P(1) \wedge P(2) \wedge P(3) \wedge P(4)$
  - $T \wedge T \wedge T \wedge F$ which is false
- So, P(4) is a counterexample and $\forall x\, P(x)$ is false.

The existential quantifier

- The existential quantifier is used to assert a **property** of **at least one** value of a variable in a **domain**.

- The existential quantification of P(x) is
"There exists an element x in the domain such that P(x).",
denoted by $\exists x\, P(x)$

  ← The existential quantifier

- The existential quantifier
  - There exists …
  - There is …
  - For some …
  - For at least one …

Ex: Let P(x): x > 3

The existential quantifier of P(x) is in the domain of integers:

$$\exists x\, P(x) \quad (x \text{ is an integer})$$

$$\exists x\, (x > 3) \quad (x \text{ is an integer})$$

- ∃x P(x)
  - When true?
    - There is an x for which P(x) is true.
      (find a value of x that P(x) is true.)
  - When false?
    - P(x) is false for every x.

P(x): x > 3.

What is the truth value of ∃x P(x) in the domain of real numbers?

Solution:

- Check if P(x) is true for some real numbers
  - "x > 3" is true when x = 4.
    So, the truth value of ∃x P(x) is true.

Q(x): x = x+1.

What is the truth value of ∃x Q(x) in the domain of real numbers?

Solution:

- Check if Q(x) is false for all real numbers
  - "x = x+1" is false for all real numbers.

    So, the truth value of ∃x Q(x) is false.

- ∃x P(x) in the domain D
- If D can be listed as $x_1, x_2, \ldots, x_n$.

∃x P(x) in the domain D is the same as

$$\mathbf{P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)}$$

P(x): $x^2 > 10$.

What is the truth value of ∃x P(x) in the domain of positive integers not exceeding 4?

Solution:

- List the domain
  - Domain is 1, 2, 3, 4.
- Find the equivalent disjunction and its truth value
  - P(1) v P(2) v P(3) v P(4)
  - F v F v F v T which is true.
- So, ∃x P(x) is true.

Quantifiers (review)

Statement	When True?	When False?
$\forall x\, P(x)$	P(x) is true for every x.	There is an x for which P(x) is false.
$\exists x\, P(x)$	There is an x for which P(x) is true.	P(x) is false for every x.

Translating from English into logical expression (example)

Express the following statement using predicates and quantifiers?

"Every student in this class has studied calculus."

Solution:

- Determine individual propositional function
  - P(x): x has studied calculus.
- Translate the sentence into logical expression
  - $\forall x\, P(x)$ domain: students in class

Express the following statement using predicates and quantifiers?

"Some student in this class has visited Mexico."

Solution:

- Determine individual propositional function
  - P(x): x has visited Mexico.
- Translate the sentence into logical expression
  - $\exists x\, P(x)$ domain: students in class

Express the following statement using predicates and quantifiers?

"Every student in this class has visited either the US or Mexico."

Solution:

- Determine individual propositional functions
  - P(x): x has visited the US.
  - Q(x): x has visited Mexico.
- Translate the sentence into logical expression
  - $\forall x\, (P(x) \vee Q(x))$ domain: students in class

Quantifiers with restricted domains

- What do the following statements mean for the domain of real numbers?

$$\forall x < 0, x^2 > 0 \quad \text{same as} \quad \forall x(x < 0 \to x^2 > 0)$$

$$\forall y \neq 0, y^3 \neq 0 \quad \text{same as} \quad \forall y(y \neq 0 \to y^3 \neq 0)$$

$$\exists z > 0, z^2 = 2 \quad \text{same as} \quad \exists z(z > 0 \wedge z^2 = 2)$$

Binding variables

- Variable
  - Bound
    - Quantifiers
  - Free
    - Not bound
- Turn a propositional function into a proposition
  - All variables must be **bound.**

$\exists x\ (x+y=1)$.
Is it a proposition?

Solution:

- Check if any variable is free
  - Variable x
    - bound
  - Variable y
    - Free
  - Since variable y is free, it is not a proposition.

Negating quantified expression

- Assume $\forall x\, P(x)$ is:

"Every student has taken a course in calculus."

- $\neg (\forall x\, P(x))$ is:

"It is not the case that every student has taken a course in calculus."

"There is a student who has not taken a course in calculus."

$\exists x\, \neg P(x)$

$$\neg (\forall x\, P(x)) \equiv \exists x\, \neg P(x)$$

Negating quantified expression (example)

What is the negation of $\forall x\, (x^2 > x)$?

Solution:

$\neg(\ \forall x\, (x^2 > x)\ ) \equiv$

$\exists x\, \neg(x^2 > x) \equiv$

$\exists x\, (x^2 \leq x)$

What is the negation of $\forall x\, (x^2 > x)$?

Solution:

$\neg(\ \forall x\, (x^2 > x)\ ) \equiv$

$\exists x\, \neg(x^2 > x) \equiv$

$\exists x\, (x^2 \leq x)$

De Morgan's laws for quantifiers

Negation	Equivalent st.	When true?	When false?
$\neg \forall x\, P(x)$	$\exists x\, \neg P(x)$	There is an x that P(x) is false.	For all x P(x) is true.
$\neg \exists x\, P(x)$	$\forall x\, \neg P(x)$	For all x P(x) is false.	There is an x that P(x) is true.

- $\forall x\, P(x)$ in the domain D
- If D can be listed as $x_1, x_2, \ldots, x_n$.
- $\forall x\, P(x) \equiv$
  $P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)$
- $\neg \forall x\, P(x) \equiv$
  $\neg(P(x_1) \wedge P(x_2) \wedge \ldots \wedge P(x_n)) \equiv$
  $\neg P(x_1) \vee \neg P(x_2) \vee \ldots \vee \neg P(x_n))$

- $\exists x\, P(x)$ in the domain D
- If D can be listed as $x_1, x_2, \ldots, x_n$.
- $\exists x\, P(x) \equiv$
  $P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)$
- $\neg \exists x\, P(x) \equiv$
  $\neg(P(x_1) \vee P(x_2) \vee \ldots \vee P(x_n)) \equiv$
  $\neg P(x_1) \wedge \neg P(x_2) \wedge \ldots \wedge \neg P(x_n))$

Logical Equivalences Involving Quantifiers

Statements involving predicates and quantifiers are *logically equivalent* **iff** they have the same truth value for all applications and for all domains of discourse.

$$\forall x(P(x) \wedge Q(x)) \equiv \forall x(P(x)) \wedge \forall x(Q(x))$$

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

$$\neg \exists x P(x) \equiv \forall x \neg P(x)$$

$$\neg \forall x(P(x) \rightarrow Q(x)) \equiv \exists x(P(x) \wedge \neg Q(x))$$

De Morgan's Laws for Quantifiers

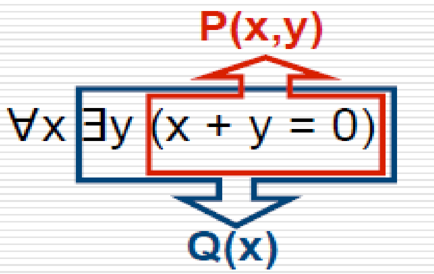
Show that $\neg\forall x\,[P(x) \rightarrow Q(x)]$ and $\exists x\,[P(x) \wedge \neg Q(x)]$ are logically equivalent

$$
\begin{aligned}
& \neg\forall x\,[P(x) \rightarrow Q(x)] & \\
\equiv\ & \exists x\,\neg[P(x) \rightarrow Q(x)] & \textit{De Morgan's} \\
\equiv\ & \exists x\,\neg[\neg P(x) \vee Q(x)] & \textit{Implication definition} \\
\equiv\ & \exists x\,[P(x) \wedge \neg Q(x)] & \textit{De Morgan's}
\end{aligned}
$$

Nested quantifiers

Two quantifiers are nested if one is within the scope of the other.

Example:



$\forall x\ Q(x)$

$Q(x)$ is $\exists y\ P(x,y)$

$P(x,y)$ is $(x + y = 0)$

Translate the following statement into English.

$\forall x\ \forall y\ (x + y = y + x)$

Domain: real numbers

Solution:

For all real numbers x and y, $x + y = y + x$.

Translate the following statement into English.

$\forall x\ \exists y\ (x = -y)$

Domain: real numbers

Solution:

For every real number x, there is a real number y such that $x = -y$.

Multiple quantifiers

- You can have multiple quantifiers on a statement

- $\forall x \exists y\ P(x, y)$
  - “For all x, there exists a y such that P(x,y)”
  - Example: $\forall x \exists y\ (x+y == 0)$

- $\exists x \forall y\ P(x,y)$
  - There exists an x such that for all y P(x,y) is true”
  - Example: $\exists x \forall y\ (x*y == 0)$

Meanings of multiple quantifiers

- $\forall x \forall y\ P(x,y)$ — P(x,y) true for all x, y pairs.
- $\exists x \exists y\ P(x,y)$ — P(x,y) true for at least one x, y pair.
- $\forall x \exists y\ P(x,y)$ — For every value of x we can find a (possibly different) y so that P(x,y) is true.
- $\exists x \forall y\ P(x,y)$ — There is at least one x for which P(x,y) is always true.

Suppose P(x,y) = “x’s favorite class is y.”

quantification order is not commutative.

The order of quantifiers

The order of nested **universal** quantifiers in a statement *without* other quantifiers can be changed without changing the meaning of the quantified statement.

The order of nested **existential** quantifiers in a statement *without* other quantifiers can be changed without changing the meaning of the quantified statement.

Common Quantifier Reversal:

$$\forall x.\ \forall y.q(x,y) \Leftrightarrow \forall y.\ \forall x.q(x,y)$$

$$\exists x.\ \exists y.q(x,y) \Leftrightarrow \exists y.\ \exists x.q(x,y)$$

Assume P(x,y) is (xy = yx).
Translate the following statement into English.
$\forall x\ \forall y\ P(x,y)$ domain: real numbers

Solution:
For all real numbers x, for all real numbers y, xy = yx.

Assume P(x,y) is (xy = 6).
Translate the following statement into English.
$\exists x\ \exists y\ P(x,y)$ domain: integers

Solution:
There is an integer x for which there is an integer y that xy = 6.

The **order** of nested existential and universal quantifiers in a statement is important.

- $\exists x \forall y$ and $\forall x \exists y$ are not equivalent!
- $\exists x \forall y\ P(x,y)$
  - $P(x,y) = (x+y == 0)$ is false
- $\forall x \exists y\ P(x,y)$
  - $P(x,y) = (x+y == 0)$ is true

Assume $P(x,y)$ is $(x + y = 10)$.

$\forall x\ \exists y\ P(x,y)$ domain: real numbers
For all real numbers x there is a real number y such that $x + y = 10$.
True $(y = 10 - x)$

$\exists y\ \forall x\ P(x,y)$ domain: real numbers
There is a real number y such that for all real numbers x, $x + y = 10$.
False
So, $\forall x\ \exists y\ P(x,y)$ and $\exists y\ \forall x\ P(x,y)$ are not logically equivalent.

Assume $P(x,y,z)$ is $(x + y = z)$.

$\forall x\ \forall y\ \exists z\ P(x,y,z)$ domain: real numbers
For all real numbers x and y there is a real number z such that $x + y = z$.
True

$\exists z\ \forall x\ \forall y\ P(x,y,z)$ domain: real numbers
There is a real number z such that for all real numbers x and y $x + y = z$.
False
So, $\forall x\ \forall y\ \exists z\ P(x,y,z)$ and $\exists z\ \forall x\ \forall y\ P(x,y,z)$ are not logically equivalent.

Nested quantifiers (example)

Translate the following statement into a logical expression.
"The sum of two positive integers is always positive."

Solution:

- Translate it to a logical expression
  "For all integers x, y, if x and y are positive, then x+y is positive."

  $\forall x \, \forall y \, ((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$ domain: integers

  $\forall x \, \forall y \, (x + y > 0)$ domain: positive integers

"Every real number except zero has a multiplicative inverse."

A multiplicative inverse of a real number x is a real number y such that xy = 1.

Solution:

- Translate it to a logical expression
  "For every real number x, if x ≠ 0, then there is a real number y such that xy = 1."

  $\forall x \, ((x \neq 0) \rightarrow \exists y \, (xy = 1))$ domain: real numbers

Statement	True when…	False when…
$\forall x \forall y \, P(x,y)$	$P(x,y)$ is true for every pair x,y	There is at least one *pair* x,y for which $P(x,y)$ is false
$\forall x \exists y \, P(x,y)$	For every x, there is a y for which $P(x,y)$ is true	There is an x for which $P(x,y)$ is false for every y
$\exists x \forall y \, P(x,y)$	There is an x for which $P(x,y)$ is true for every y	For every x, there is a y for which $P(x,y)$ is false
$\exists x \exists y P(x,y)$	There is at least one pair x,y for which $P(x,y)$ is true	$P(x,y)$ is false for every pair x,y

Negating nested quantifiers (example)

What is the negation of the following statement?

$\forall x \exists y (x = -y)$

Solution:

$\neg \forall x P(x)$ $\quad$ $P(x) = \exists y (x = -y)$

$\exists x \neg P(x)$

$\exists x (\neg \exists y (x = -y))$

$\exists x (\forall y \neg (x = -y))$

$\exists x \forall y (x \neq -y)$

- Rewrite these statements so that the negations only appear within the predicates

α) $\neg\exists y\exists x P(x,y)$

- $\forall y\neg\exists x P(x,y)$
- $\forall y\forall x \neg P(x,y)$

α) $\neg\forall x\exists y P(x,y)$

- $\exists x\neg\exists y P(x,y)$
- $\exists x\forall y \neg P(x,y)$

α) $\neg\exists y (Q(y) \wedge \forall x \neg R(x,y))$

- $\forall y \neg(Q(y) \wedge \forall x \neg R(x,y))$
- $\forall y (\neg Q(y) \vee \neg(\forall x \neg R(x,y)))$
- $\forall y (\neg Q(y) \vee \exists x R(x,y))$

α)

- $\neg(\forall x\exists y P(x,y) \vee \forall x\exists y Q(x,y))$
- $\neg\forall x\exists y P(x,y) \wedge \neg\forall x\exists y Q(x,y)$
- $\exists x\neg\exists y P(x,y) \wedge \exists x\neg\exists y Q(x,y)$
- $\exists x\forall y \neg P(x,y) \wedge \exists x\forall y \neg Q(x,y)$

Translating between English and quantifiers

- The product of two negative integers is positive
  - $\forall x \forall y\ ((x<0) \land (y<0) \rightarrow (xy > 0))$
  - Why conditional instead of and?
- The average of two positive integers is positive
  - $\forall x \forall y\ ((x>0) \land (y>0) \rightarrow ((x+y)/2 > 0))$
- The difference of two negative integers is not necessarily negative
  - $\exists x \exists y\ ((x<0) \land (y<0) \land (x-y \geq 0))$
  - Why and instead of conditional?
- The absolute value of the sum of two integers does not exceed the sum of the absolute values of these integers
  - $\forall x \forall y\ (|x+y| \leq |x| + |y|)$

Translating between English and quantifiers

- $\exists x \forall y\ (x+y = y)$
  - There exists an additive identity for all real numbers
- $\forall x \forall y\ (((x \geq 0) \land (y<0)) \rightarrow (x-y > 0))$
  - A non-negative number minus a negative number is greater than zero
- $\exists x \exists y\ (((x \leq 0) \land (y \leq 0)) \land (x-y > 0))$
  - The difference between two non-positive numbers is not necessarily non-positive (i.e. can be positive)
- $\forall x \forall y\ (((x \neq 0) \land (y \neq 0)) \leftrightarrow (xy \neq 0))$
  - The product of two non-zero numbers is non-zero if and only if both factors are non-zero

Rules of Inference :

Simple arguments can be used as building blocks to construct more complicated valid arguments. Certain simple arguments that have been established as valid are very important in terms of their usage. These arguments are called Rules of

Definition

An *argument* in propositional logic is sequence of propositions. All but the final proposition are called *premises* and the final proposition is called the *conclusion*. An argument is *valid* if the truth of all its premises implies that the conclusion is true.

An *argument form* in propositional logic is a sequence of compound propositions involving propositional variables. An argument form is *valid* if no matter which propositions are substituted for the propositional variables in its premises, if the premises are all true, then the conclusion is true.

$$\begin{array}{l} \text{First Premise} \\ \text{Second Premise} \\ \text{Third Premise} \\ . \\ . \\ \text{Nth Premise} \\ \hline \therefore \textit{Conclusion} \end{array}$$

Modus Ponens

If P and $P \to Q$ are two premises, we can use Modus Ponens to derive Q.

$$\begin{array}{c} P \to Q \\ P \\ \hline \therefore Q \end{array}$$

Example

"If you have a password, then you can log on to facebook", $P \to Q$

"You have a password", P

Therefore – "You can log on to facebook"

Modus Tollens

If $P \to Q$ and $\neg Q$ are two premises, we can use Modus Tollens to derive $\neg P$.

$$\begin{array}{c} P \to Q \\ \neg Q \\ \hline \therefore \neg P \end{array}$$

Example

"If you have a password, then you can log on to facebook", $P \to Q$

"You cannot log on to facebook", $\neg Q$

Therefore – "You do not have a password "

Conjunction

If P and Q are two premises, we can use Conjunction rule to derive $P \wedge Q$.

$$\begin{array}{c} P \\ Q \\ \hline \therefore P \wedge Q \end{array}$$

Example

Let P – "He studies very hard"

Let Q – "He is the best boy in the class"

Therefore – "He studies very hard and he is the best boy in the class"

TABLE 1 **Rules of Inference.**

Rule of Inference	*Tautology*	*Name*
p $p \to q$ $\therefore q$	$(p \wedge (p \to q)) \to q$	Modus ponens
$\neg q$ $p \to q$ $\therefore \neg p$	$(\neg q \wedge (p \to q)) \to \neg p$	Modus tollens
$p \to q$ $q \to r$ $\therefore p \to r$	$((p \to q) \wedge (q \to r)) \to (p \to r)$	Hypothetical syllogism
$p \vee q$ $\neg p$ $\therefore q$	$((p \vee q) \wedge \neg p) \to q$	Disjunctive syllogism
p $\therefore p \vee q$	$p \to (p \vee q)$	Addition
$p \wedge q$ $\therefore p$	$(p \wedge q) \to p$	Simplification
p q $\therefore p \wedge q$	$((p) \wedge (q)) \to (p \wedge q)$	Conjunction
$p \vee q$ $\neg p \vee r$ $\therefore q \vee r$	$((p \vee q) \wedge (\neg p \vee r)) \to (q \vee r)$	Resolution

Ex: Determine whether the argument is valid and whether the conclusion must be true

- If $\sqrt{2} > \frac{3}{2}$ then $(\sqrt{2})^2 > (\frac{3}{2})^2$. We know that $\sqrt{2} > \frac{3}{2}$. Therefore, $(\sqrt{2})^2 = 2 > (\frac{3}{2})^2 = \frac{9}{4}$.
- Is the argument valid?
- Does the conclusion must be true?

$$\sqrt{2} > \frac{3}{2} \to (\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$$

$$\sqrt{2} > \frac{3}{2}$$

$$\therefore 2 > \frac{9}{4}$$

The argument is valid as it is constructed using modus ponens
But one of the premises is false (p is false)
So, we cannot derive the conclusion

A valid argument can lead to an incorrect conclusion if one of its premises is wrong/false!