IMAGES AND INVERSE IMAGES OF FUNCTIONS

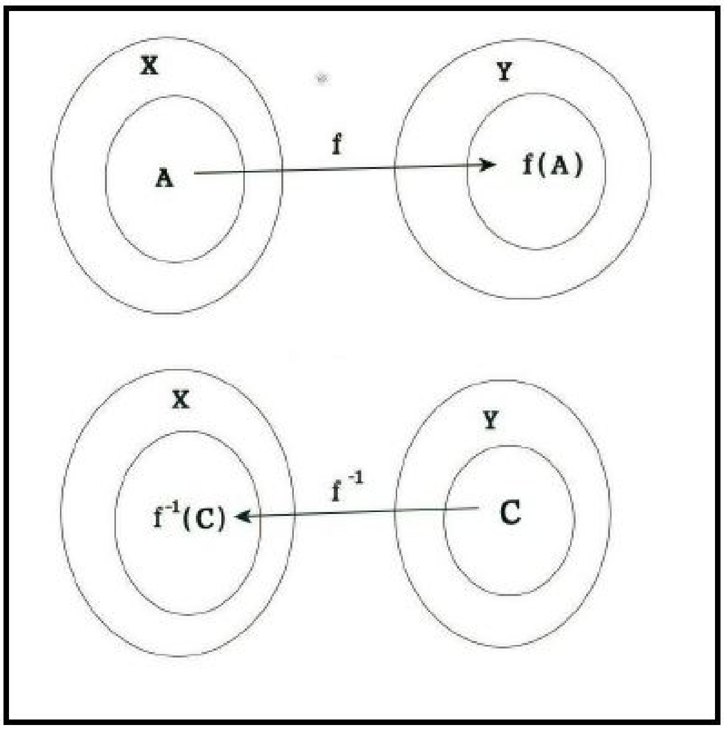
Definition 1. Let $f : X \to Y$ with $A \subseteq X$ and $C \subseteq Y$.
As x moves about the set A, the set of values $f(x)$ defines the **image of** A notated as

$$f(A) = \{f(x) : x \ \in A\}.$$

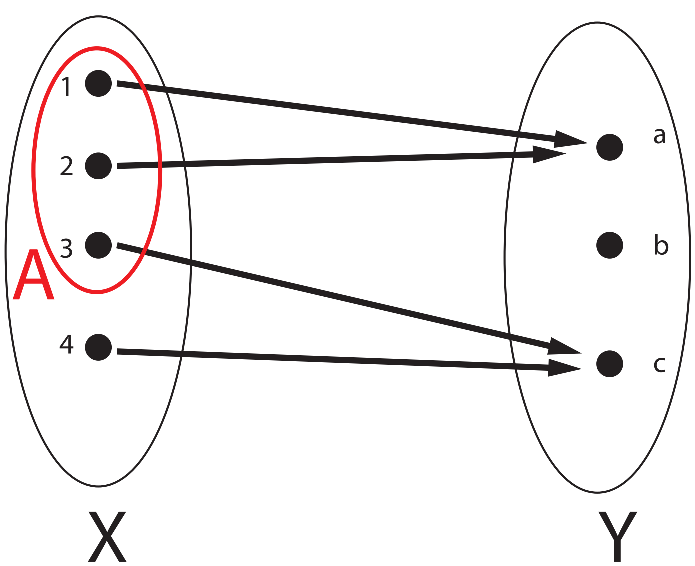
Also, we define the **inverse image of** C as the set

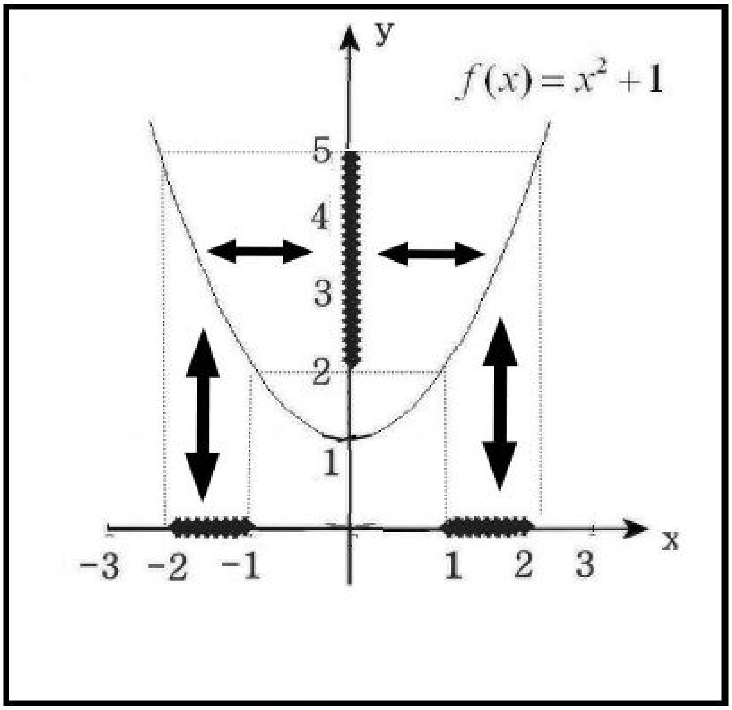
$$f^{-1}(C) = \{x \in X : f(x) \in C\}.$$

Note that $f^{-1}(C)$ is well defined whether or not the function f has an inverse.



Example 1. Let $X = \{1, 2, 3, 4\}$ and $Y = \{a, b, c\}$. Define function $f : X \to Y$ by $f(1) = a$, $f(2) = a$, $f(3) = c$, and $f(4) = c$. See figure below. Letting $A = \{1, 2, 3\}$ we have $f(A) = \{a, c\}$. Also, $f^{-1}(\{b, c\}) = \{3, 4\}$, $f^{-1}(\{a, c\}) = \{1, 2, 3, 4\}$, and $f^{-1}(\{b\}) = \emptyset$.





$f^{-1}([2,5]) = [-2,-1] \cup [1,2]$

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 1$. See above figure.

Then we have

(a) $f(\{-1, 1\}) = \{2\}$ since both $f(-1) = f(1) = 2$.
(b) $f([-2, 2]) = [1, 5]$
(c) $f([-2, 3]) = [1, 10]$
(d) $f^{-1}(\{1, 5, 10\}) = \{-3, -2, 0, 2, 3\}$
(e) $f^{-1}([0, 1]) = \{0\}$
(f) $f^{-1}([2, 5]) = [-2, -1] \cup [1, 2]$

Exercise 1. Let $f : A \to B$ be a function and let $C \subseteq A$. Prove that C is a subset of $f^{-1}(f(C))$. Also, show by example that equality need not hold.

IMAGES OF INTERSECTIONS AND UNIONS

Theorem 1. *If $f : X \to Y$ and A and B are subsets of the domain X, then*

$$f(A \cap B) \subseteq f(A) \cap f(B).$$

See figure below for illustration.

Proof. Let $y \in f(A \cap B)$. Hence, there exists $x \in A \cap B$ such that $f(x) = y$. Note that $x \in A$ and $x \in B$. It follows that $y = f(x) \in f(A)$ and $y = f(x) \in f(B)$. Thus $y \in f(A) \cap f(B)$. □

To see that equality above does not necessarily follow we consider the following example.

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$. Using $A = [-1, 0]$ and $B = [0, 1]$ we have $A \cap B = \{0\}$ and so $f(A \cap B) = f(\{0\}) = \{0\}$, whereas $f(A) \cap f(B) = [0, 1] \cap [0, 1] = [0, 1]$. Thus $f(A \cap B) \neq f(A) \cap f(B)$.

The above example shows that intersections of sets are not always preserved under the image of a function. However, the following theorem shows that intersections of sets are preserved under the image of ***one-to-one*** functions.

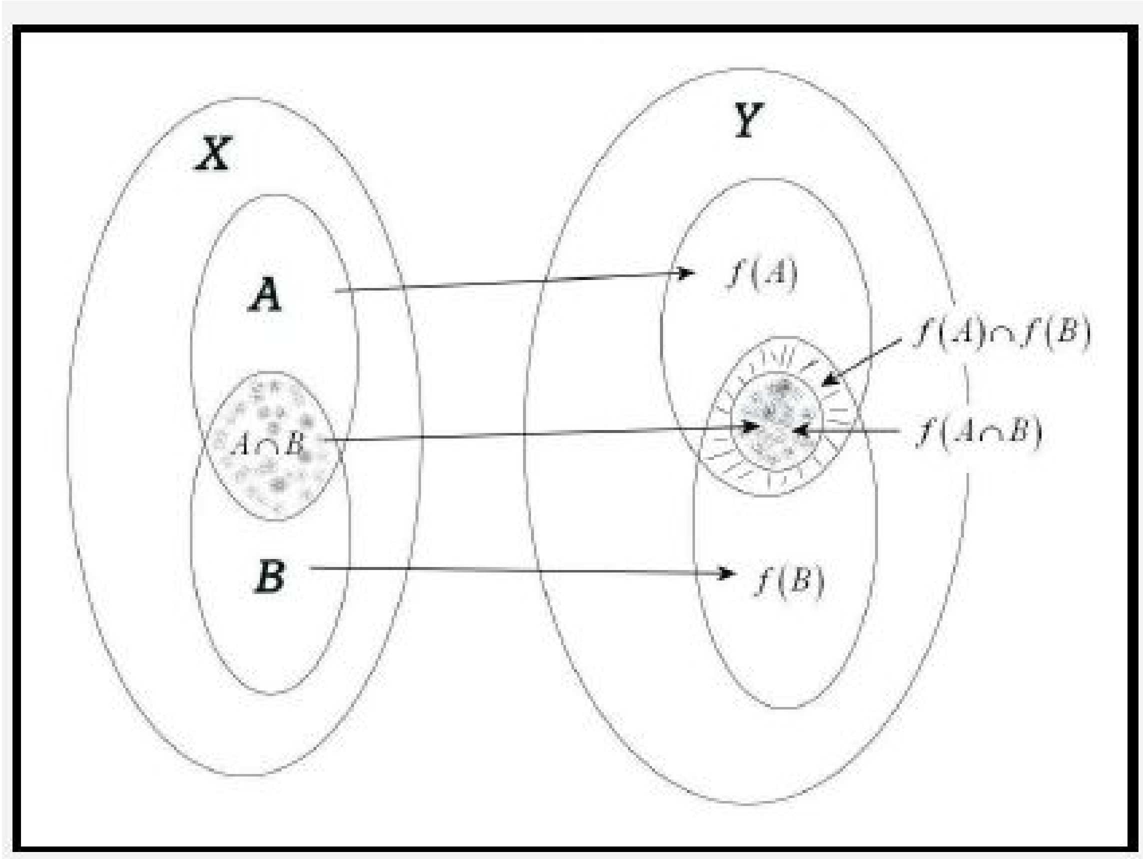


Illustration of Theorem 1.

Theorem 2. *Let $f : X \to Y$ be a one-to-one function and let A and B be subsets of the domain X. Then*

$$f(A \cap B) = f(A) \cap f(B).$$

Proof. Considering Theorem 1 it suffices to show $f(A) \cap f(B) \subseteq f(A \cap B)$. Let $y \in f(A) \cap f(B)$, i.e., $y \in f(A)$ and $y \in f(B)$. Hence, there exists $x_1 \in A$ such that $f(x_1) = y$ and there exists $x_2 \in B$ such that $f(x_2) = y$. Since $f(x_1) = y = f(x_2)$ and f is one-to-one, we have $x_1 = x_2$ and so $x_1 \in A \cap B$. This implies $y = f(x_1) \in f(A \cap B)$. □

Although intersections are not always preserved under image of a function (though it is if the function is one-to-one), we now show that unions are always preserved under this action (whether the function is one-to-one or not).

Theorem 3. *Let $f : X \to Y$ be a function and let A and B be subsets of the domain X. Then*

$$f(A \cup B) = f(A) \cup f(B).$$

Proof. Let $y \in f(A \cup B)$, i.e., there exists $x \in A \cup B$ such that $y = f(x)$. Hence, $x \in A$ or $x \in B$ and so $y = f(x) \in f(A)$ or $y = f(x) \in f(B)$, i.e., $y \in f(A) \cup f(B)$. We have thus demonstrated that $f(A \cup B) \subseteq f(A) \cup f(B)$.

We leave it to the reader to verify (essentially by reversing the steps above) the reverse inclusion, namely, $f(A \cup B) \supseteq f(A) \cup f(B)$. □

The following result shows that under the action of inverse images of functions, both unions and intersections are preserved (whether the function is one-to-one or not).

Theorem 4. *Let $f : X \to Y$ be a function and let C and D be subsets of the codomain Y. Then*

$$(a)\ f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D),$$
$$(b)\ f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D).$$

Proof. We prove (a) by noting

$$\begin{aligned} x \in f^{-1}(C \cap D) &\Leftrightarrow f(x) \in C \cap D \\ &\Leftrightarrow f(x) \in C \text{ and } f(x) \in D \\ &\Leftrightarrow x \in f^{-1}(C) \text{ and } x \in f^{-1}(D) \\ &\Leftrightarrow x \in f^{-1}(C) \cap f^{-1}(D). \end{aligned}$$

We leave it to the reader to verify (b). □