Inequalities

Inequalities: A mathematical statement that contains $>, <, \leq$ or $\geq$ is called inequality.

Example: Solve the following linear inequalities:

1. $\mathbf{2x + 5 < 13}$

$2x + 5 - 5 < 13 - 5$

$2x < 8$ $\quad$ (÷ 2 both sides)

$x < 4$

s.s is $(-\infty, 4)$ or $\{x \in R : x < 4\}$

2. $\mathbf{4x + 3 < -9}$

$4x + 3 - 3 < -9 - 3$

$4x < -12$ (÷ 4 both sides)

$x < -3$

s.s is $(-\infty, -3)$ or $\{x \in R: x < -3\}$

3. $\mathbf{1 - 3x \geq 2x - 4}$

$-3x - 2x \geq -4 - 1$

$-5x \geq -5$ (÷ (-5) both sides)

$x \leq 1$

s.s is $(-\infty, 1]$ or $\{x \in R: x \leq 1\}$

4. $\mathbf{3 - 4x \leq -5x + 2 < 9x + 8}$

$3 - 4x \leq -5x + 2$ and $-5x + 2 < 9x + 8$

$-4x + 5\text{x} \leq 2 - 3$ and $-5x - 9x < 8 - 2$

$x \leq -1$ and $-14x < 6$ (÷ (-14) both sides)

$x \leq -1$ and $x > \frac{-3}{7}$

s.s is $(-\infty, -1] \cap \left(\frac{-3}{7}, \infty\right) = \emptyset$

5. $\mathbf{\frac{-x}{2} \leq 3x - 5}$ ($\times$ 2 both sides)

$-x \leq 2(3x - 5)$

$-x \leq 6x - 10$

$-x - 6x \leq -10$

$-7x \leq -10 \Longrightarrow x \geq \frac{10}{7}$

s.s is $[\frac{10}{7}, \infty)$

Rule of inequalities:

Let a, b, c are real numbers, then

1. $a < b \Longrightarrow a \mp c < b \mp c$

2. $a < b$ and $c > 0 \Longrightarrow a.c < b.c$

3. $a < b$and $c < 0 \Longrightarrow a.c > b.c$, (special case $a < b \Longrightarrow -a > -b$)

4. $a > 0 \Longrightarrow \frac{1}{a} > 0$

5. If a and b have the same signs, and $a, b \neq 0$ then $a < b \Longrightarrow \frac{1}{a} > \frac{1}{b}$

6. If a and b have the different signs, and $a, b \neq 0$ then $a < b \Longrightarrow \frac{1}{a} < \frac{1}{b}$

7. $a.b \geq 0$, either $a, b \geq 0$ or $a, b \leq 0$

8. $a.b > 0$, either $a, b > 0$ or $a, b < 0$

9. $a.b < 0$, either $a > 0, b < 0$ or $a < 0, b > 0$

10. $\frac{a}{b} > 0$, either $a, b > 0$ or $a, b < 0$

11. $\frac{a}{b} \geq 0$, either $a \geq 0, b > 0$ or $a \leq 0, b < 0$

12. $\frac{a}{b} < 0$, either $a > 0, b < 0$ or $a < 0, b > 0$

<u>Example:</u> Find the solution set of the following:

1. $\frac{x-1}{x+3} < 0$

i. $x - 1 > 0$ and $x + 3 < 0$

$x > 1$ and $x < -3$

s.s $=(-\infty, -3) \cap (1, \infty) = \emptyset$

ii. $x - 1 < 0$ and $x + 4 > 0$

$x < 1$ and $x > -3$

s.s $=(-\infty, 1) \cap (-3, \infty) = (-3,1)$

s.s is $\emptyset \cup (-3,1) = (-3,1)$

2. $\frac{\mathbf{2x+5}}{\mathbf{5x+7}} > \mathbf{0}$

i. $2x + 5 > 0$ and $5x + 7 > 0$

$2x > -5$ and $5x > -7$

$x > \frac{-5}{2}$ and $x > \frac{-7}{5}$

s.s $= \left(\frac{-5}{2}, \infty\right) \cap \left(\frac{-7}{5}, \infty\right) = (\frac{-7}{5}, \infty)$

ii. $2x + 5 < 0$ and $5x + 7 < 0$

$2x < -5$ and $5x < -7$

$x < \frac{-5}{2}$ and $x < \frac{-7}{5}$

s.s $= \left(-\infty, \frac{-5}{2}\right) \cap \left(-\infty, \frac{-7}{5}\right) = (-\infty, \frac{-5}{2})$

s.s is $\left(-\infty, \frac{-5}{2}\right) \cup \left(\frac{-7}{5}, \infty\right) = R \setminus [\frac{-5}{2}, \frac{-7}{5}]$

3. $\frac{\mathbf{1}}{\mathbf{2x-3}} < \mathbf{1} \Rightarrow \frac{1}{2x-3} - 1 < 0$

$$\frac{1-2x+3}{2x-3} < 0$$

$$\frac{-2x+4}{2x-3} < 0$$

i. $-2x+4 > 0$ and $2x-3 < 0$

$-2x > -4$ and $2x < 3$

$x < 2$ and $x < \frac{3}{2} \Rightarrow$ s.s= $(-\infty, \frac{3}{2})$

ii. $-2x+4 < 0$ and $2x-3 > 0$

$-2x < -4$ and $2x > 3$

$x > 2$ and $x > \frac{3}{2} \Rightarrow$s.s= $(2, \infty)$

s.s is $(-\infty, \frac{3}{2}) \cup (2, \infty) = R\backslash[\frac{3}{2}, 2]$

4. $\mathbf{\frac{3}{x-4} < 0}$

since $3 > 0$, then $x - 4 < 0 \Longrightarrow x < 4$

s.s is $(-\infty, 4)$

5. $\mathbf{\frac{3}{x^2-2x+5} \leq 0}$

Since $3 > 0$

$\Longrightarrow \quad x^2 - 2x + 5 \leq 0$ $\qquad ax^2 + bx + (\frac{b}{2})^2 - (\frac{b}{2})^2 + c$

$a = 1, b = -2, c = 5$, so $(\frac{-2}{2})^2 = 1$

$x^2 - 2x + 1 - 1 + 5 \leq 0$

$\therefore (x - 1)^2 + 4 \leq 0$

Always positive $+4 \leq 0$ (×)

s.s $= \emptyset$

2. $\boldsymbol{x^2 - x - 12 \geq 0}$

$(x+3)(x-4) \geq 0$

i. $x + 3 \geq 0$ and $x - 4 \geq 0$

$x \geq -3$ and $x \geq 4$

s.s = $[-3, \infty) \cap [4, \infty) = [4, \infty)$

ii. $x + 3 \leq 0$ and $x - 4 \leq 0$

$x \leq -3$ and $x \leq 4$

s.s $= (-\infty, -3] \cap (-\infty, 4] = (-\infty, -3]$

$\therefore$ s.s is $(-\infty, -3] \cup [4, \infty) = R \backslash (-3,4)$

Example: Find the solution set of the following: **H.w**

1. Prove that $\sqrt{3}$ is an irrational number.

2. $x^2 + 3x + 10 \geq 0$

3. $x^2 - 2x + 1 \geq 0$

4. $\frac{2x+5}{5x+7} < 0$

5. $\frac{3-6x}{7x+1} \geq 0$

6. $\frac{x(x-1)}{x-2} \leq 0$

7. $\frac{x+1}{x^2-4} \geq 0$

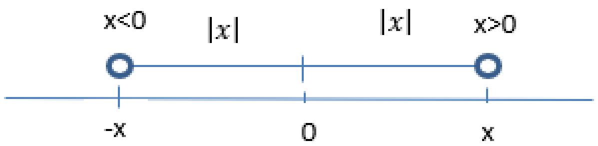
Absolute value

The **absolute value** of any real number x is defined as:

$$|x| = \begin{cases} x \,, x \geq 0 \\ -x \,, x < 0 \end{cases}$$

Example: $|-3| = 3, |5| = 5, |0| = 0$

Geometrically the absolute value of the x is the distance from 0 to x.



Properties of absolute values: Let $x, y \in R$, then

1. $|-x| = x$

2. $|x| = \sqrt{x^2}$

3. $|x \cdot y| = |x| \cdot |y|$

4. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, y \neq 0$

5. $|x + y| \leq |x| + |y|$ (the triangle inequality)

Proof 5: if $x \geq 0$ and $y \geq 0 \Longrightarrow x + y \geq 0$

$x \leq |x|, \qquad y \leq |y|$

$x + y \leq |x| + |y| \ldots \ldots \ldots \ldots (1)$

if $x \leq 0$ and $y \leq 0 \Longrightarrow x + y \leq 0$

$-x \leq |x|, \qquad -y \leq |y|$

$$-x - y \leq |x| + |y|$$

$$-(x + y) \leq |x| + |y|$$

$$x + y \geq -[|x| + |y|] \dots \dots \dots \dots (2)$$

From (1) and (2) we get

$$-[|x| + |y|] \leq x + y \leq |x| + |y| \Longrightarrow |x + y| \leq |x| + |y|$$

Remark: If D is a positive number, then

1. $|x| = D \quad \Leftrightarrow \quad$ either $x = -D$ or $x = D$
2. $|x| < D \quad \Leftrightarrow \quad -D < x < D$
3. $|x| \leq D \quad \Leftrightarrow \quad -D \leq x \leq D$
4. $|x| > D \quad \Leftrightarrow \quad$ either $x < -D$ or $x > D$

More generally,

6. $|x-a| = D \quad \Leftrightarrow \quad$ either $x = a - D$ or $x = a + D$

7. $|x-a| < D \quad \Leftrightarrow \quad \text{a} - D < x < a + D$

8. $|x-a| \leq D \quad \Leftrightarrow \quad \text{a} - D \leq x \leq a + D$

9. $|x-a| > D \quad \Leftrightarrow \quad$ either $x < a - D$ or $x > a + D$

Some other properties

1. $|x| \geq x, \ \forall x \in R \Longrightarrow$ s.s is R

2. $-|x| \leq x, \ \forall x \in R \Longrightarrow$ s.s is R

3. $|x| > x, x < 0$, s.s is $(-\infty, 0)$

4. $|x| < x,$ s.s is $\emptyset$

5. $|x| = x, x \geq 0$, s.s is $[0, \infty)$

6. $|x| > -x, \quad x > 0$

7. $|x| < -x$, s.s is $\emptyset$

8. $|x| \leq x$, s.s is $[0, \infty)$

Example: Find the solution set of the following inequalities

1. $\boldsymbol{|2x - 1| \leq 3}$

$-3 \leq 2x - 1 \leq 3$

$-3 + 1 \leq 2x - 1 + 1 \leq 3 + 1$

$-2 \leq 2x \leq 4$

$\Rightarrow -1 \leq x \leq 2$

s.s= $[-1,2]$

2. $\boldsymbol{|3+x|<1}$

$3+x>1$ or $3+x<-1$

$x>1-3$ or $x<-1-3$

$x>\ -2$ or $x<-4$

s.s= $(-\infty,-4)\cup(-2,\infty)=R\backslash[-4,-2]$

3. $\boldsymbol{|2x-1|<-3\Longrightarrow}$ s.s= $\emptyset$

4. $\boldsymbol{|2x-1|>2x-1}$

$2x-1<0$ by def $|x|>x$

$$2x<1\Longrightarrow x<\frac{1}{2}$$

s.s= $(-\infty,\frac{\mathbf{1}}{\mathbf{2}})$

5. $|\mathbf{2x-1}| > \mathbf{1-2x} \Longrightarrow |2x-1| > \; -(2x-1)$

$2x - 1 > 0$ by def $\quad |x| > \; -x$

$$2x > 1 \Longrightarrow x > \frac{1}{2}$$

s.s $= (\frac{1}{2}, \infty)$

6. $\left|\frac{\boldsymbol{x-4}}{\mathbf{5}}\right| \leq \mathbf{1}$

$\frac{|x-4|}{5} \leq 1 \qquad (* \text{ 5 both sides})$

$|x - 4| \leq 5$

$-5 \leq x - 4 \leq 5 \Longrightarrow -5 + 4 \leq x - 4 + 4 \leq 5 + 4$

$-1 \leq x \leq 9$

$\therefore$ s.s$= [-1{,}9]$

7. $\left|\frac{\mathbf{-3}}{\mathbf{2-1}}\right| > \mathbf{4} \Longrightarrow \frac{3}{|2-x|} > 4 \Longrightarrow 3 > 4|2-x|, \qquad |-3| = 3$

$$|2-x| < \frac{3}{4} \Longrightarrow \frac{-3}{4} < 2-x < \frac{3}{4}$$

$$\frac{-3}{4} - 2 < 2 - 2 - x < \frac{3}{4} - 2$$

$$\frac{-11}{4} < -x < \frac{-5}{4} \Longrightarrow \frac{11}{4} > x > \frac{-5}{4}$$

$$\frac{-5}{4} < x < \frac{11}{4}$$

Therefor s.s $= (\frac{-5}{4}, \frac{11}{4})$

Example: Solve the inequalities:

1. $|\mathbf{2x+5}| = \mathbf{3}$

$2x+5=-3$ or $2x+5=3$

$2x+5-5=-3-5$ or $2x+5-5=3-5$

$x=-4$ or $x=-1$

The solutions are $x=-4$ and $x=-1$

2. $|\mathbf{3x-2}| \leq \mathbf{1} \Leftrightarrow -1 \leq 3x-2 \leq 1$

$-1+2 \leq 3x-2+2 \leq 1+2$

$1 \leq 3x \leq 3 \quad (\div 3)$

$\frac{1}{3} \leq x \leq 1$

s.s= $[\frac{1}{3}, 1]$

Example: Find the solution set of the following inequalities

1. $\left|\boldsymbol{x^2 - 2x + 4}\right| \geq \boldsymbol{x^2 - 2|x + 1| + 6}$

$|x^2 - 2x + 4| = x^2 - 2x + 4$

$x^2 - 2x + 4 \geq x^2 - 2|x + 1| + 6$

$-2x + 4 - 6 \geq -2|x + 1|$

$-2(x + 1) \geq -2|x + 1| \Rightarrow x + 1 \leq |x + 1|$

$|x + 1| \geq x + 1$

$\therefore$ s.s= R by def. $|x| \geq x$

2. $\boldsymbol{|5 - 2x| < -3}$

Since $|5 - 2x|$ is always positive

$\therefore$ s.s= $\emptyset$

3. $\boldsymbol{|4x + 5| > 7}$ (negation)

$|4x + 5| \le 7$

$-7 \le 4x + 5 \le 7$

$-7 - 5 \le 4x + 5 - 5 \le 7 - 5$

$-12 \le 4x \le 2$ $(\div 4)$

$-3 \le x \le \frac{1}{2} \Longrightarrow$ s.s= $[-3, \frac{1}{2}]$

$\therefore$ s.s of $|4x + 5| > 7$ is $R \backslash [-3, \frac{1}{2}]$

another way $|\mathbf{4x+5}| > \mathbf{7}$

$4x+5 < -7$ or $4x+5 > 7$

$4x+5-5 < -7-5$ or $4x+5-5 > 7-5$

$4x < -12$ or $4x > 2$

$\Longrightarrow x < -3$ or $x > \frac{1}{2}$

s.s= $(-\infty,-3) \cup \left(\frac{1}{2},\infty\right) = R\backslash[-3,\frac{1}{2}]$

4. $\mathbf{1 \leq |x-1| \leq 3}$

$|x-1| \geq 1$ or $|x-1| \leq 3$

$x-1 \geq 1$, $x-1 \leq -1$ $\quad$ $-3 \leq x-1 \leq 3$

$x \geq 2$ $\quad$ $x \leq 0$ $\quad$ $-2 \leq x \leq 4$

$-2 \leq x \leq 0$ or $2 \leq x \leq 4$

$\therefore$ s.s= $[-2,0] \cup [2,4]$

5. $|\boldsymbol{x}| > |\boldsymbol{x} - \mathbf{1}|$

$$|x| = \begin{cases} x\ , & x \geq 0 \\ -x\ , & x < 0 \end{cases} \quad , \quad |x-1| = \begin{cases} x-1\ , & x \geq 1 \\ -(x-1)\ , & x < 1 \end{cases}$$

i. $x < 0$

$-x > -(x-1) \Rightarrow -x > -x + 1 \Rightarrow 0 > 1$ no solution

s.s= $\emptyset$

ii. $0 \leq x < 1$

$x > -x + 1 \Rightarrow x + x > 1$

$2x > 1 \Rightarrow \quad x > \frac{1}{2}, \qquad (\frac{1}{2}, \infty)$

iii. $x \geq 1$

$x > \ x - 1$

$0 > -1$ always true, $[1, \infty)$

s.s= $\emptyset \cup \left(\frac{1}{2}, 1\right) \cup [1, \infty)$