H.W ① $\int \sin^{-1} \frac{1}{\sqrt{1-x^2}}\, dx$ ② $\int \sqrt{\cot x}\, \csc^2 x\, dx$

③ $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\, dx$

Techniques of Integration

① u substitution Method

Let $u = g(x)$ be a differential function whose range is an interval I and f is continuous on I.

Then $\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du$

$\left(\frac{du}{dx} = g'(x) \Rightarrow du = g'(x)\, dx \right)$

e.g ① $\int \cos(5x-2)\, dx$, let $u = 5x-2$

$\Rightarrow du = 5\, dx \Rightarrow dx = \frac{1}{5} du$

$\Rightarrow \int \cos(u) (\frac{1}{5}) du = \frac{1}{5} \int \cos u\, du = \frac{1}{5} \sin u + c$

$= \frac{1}{5} \sin(5x-2) + c$

② $\int \frac{\sin^2(\frac{1}{x})}{x^2}\, dx$, let $u = \frac{1}{x} \Rightarrow du = \frac{-1}{x^2}$

$du = \frac{-1}{x^2}\, dx$

$\Rightarrow$

$$\int \frac{\sin^2(\frac{1}{x})}{x^2}\,dx = -\int \sin^2 u\,du = \int \frac{(\cos 2u - 1)}{2}\,du$$

$$= \frac{1}{2}\int(\cos 2u - 1)\,du \qquad \sin^2 x = \frac{1-\cos 2x}{2}$$

$$= \frac{1}{2}\left(\frac{1}{2}\sin 2u - u\right) + c$$

$$= \frac{1}{4}\sin 2u - \frac{1}{2}u + c \quad (\text{We have } u = \tfrac{1}{x})$$

$$= \frac{1}{4}\sin(2/x) - \frac{1}{2}\left(\frac{1}{x}\right) + c$$

$$= \frac{1}{4}\sin(2/x) - \frac{1}{2x} + c$$

(3) $\int \frac{dx}{\sqrt{1+4x}}$, let $u = 1+4x \Rightarrow du = 4dx \Rightarrow dx = \frac{1}{4}du$

$$\Rightarrow \int \frac{1}{\sqrt{1+4x}}\,dx = \int \frac{\frac{du}{4}}{\sqrt{u}} = \frac{1}{4}\int \frac{du}{\sqrt{u}}$$

$$= \frac{1}{4}\int u^{-\frac{1}{2}}\,du = \frac{1}{4}\cdot\frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c = \frac{1}{4}(2)\sqrt{u} + c$$

$$= \frac{1}{2}\sqrt{u} + c = \frac{1}{2}\sqrt{1+4x} + c$$

H.W (1) $\int e^{\frac{x}{2}}\,dx$ (2) $\int \frac{x}{\sqrt{1+x^2}}\,dx$

② Integration by parts

let u & V be differentiable of x, the product rule say

$$\frac{d}{dx}\left(u(x).V(x)\right) = u(x).V'(x) + V(x).u'(x)$$

$$\int \frac{d}{dx}(u.V)\,dx = \int (u.V' + V.u')\,dx$$

$$u.V = \int u V'\,dx + \int V.u'\,dx \qquad V' = \frac{dv}{dx}.dx$$

$$u.V = \int u\,dv + \int V\,du \qquad u' = \frac{du}{dx}.dx$$

$$\Rightarrow \int u\,dv = u.V - \int V\,du$$

e.g

① $\int \ln x\,dx$, $u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$dv = dx \Rightarrow V = x$

$$\int \ln x\,dx = x\ln x - \int x\,\frac{1}{x}\,dx = x\ln x - x + c$$

② $\int x \sin x\,dx$

$u = x$, $du = dx$

$dv = \sin x\,dx \Rightarrow V = -\cos x$

$$\therefore \int u\,dv = u.v - \int v\,du$$

$$\int x\sin x\,dx = x.(-\cos x) - \int (-\cos x)\,dx$$

$$= -x\cos x + \int \cos x\,dx$$

$$= -x\cos x + \sin x + c$$

③ $\int x e^{-x}\,dx$

$u = x \Rightarrow du = dx$

$dv = e^{-x}\,dx \Rightarrow v = \frac{e^{-x}}{-1} = -e^{-x}$

$$\int x e^{-x}\,dx = x.(-e^{-x}) - \int -e^{-x}\,dx$$

$$= -x e^{-x} + \int e^{-x}\,dx$$

$$= -x e^{-x} + \frac{e^{-x}}{-1} + c$$

$$= -x e^{-x} - e^{-x} + c$$

$$= -e^{-x}(x+1) + c$$

④ $\int x\ln x\,dx$

$u = \ln x \Rightarrow du = \frac{1}{x}\,dx$

$dv = x\,dx \Rightarrow v = \frac{x^2}{2}$

$$\int x\ln x\,dx = \ln x \cdot \left(\frac{x^2}{2}\right) - \int \left(\frac{x^2}{2}\right)\frac{1}{x}\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$$

$$= \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + c$$

$$= \frac{x^2}{2}\left(\ln x - \frac{1}{2}\right) + c$$

$\int \sin x \ln \cos x \, dx$

let $u = \ln \cos x \Rightarrow du = \frac{1}{\cos x}(-\sin x)\,dx$

$dv = \sin x \, dx \Rightarrow v = -\cos x$

$\Rightarrow \int \sin x \ln \cos x \, dx = -\cos x \ln \cos x - \int -\cos x \left(\frac{-\sin x}{\cos x}\right) dx$

$= -\cos x \ln \cos x - \int \sin x \, dx$

$= -\cos x \ln \cos x + \cos x + c$

H.W ① $\int \frac{\ln x}{x} \, dx$

② $\int \tan^{-1} x \, dx$

③ Trignometric integrals (power of Sine & Cosine)

① If the power of cosine is odd (i.e $n = 2k+1$) Save one Cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors interms of sine

$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \underbrace{\cos^{2k} x}_{(\cos^2 x)^k} (\cos x)^1 \, dx$

$= \int \sin^m x (1 - \sin^2 x)^k \cos x \, dx$

e.g ① $\int \sin^4 x \cos^3 x \, dx$ $\qquad m=4, \; n=3$

$= \int \sin^4 x \cos^2 x \cos x \, dx$

$= \int \sin^4 x (1 - \sin^2 x) \cos x \, dx$

$= \int \sin^4 x \cos x \, dx - \int \sin^6 x \cos x \, dx$

$= \int (\sin x)^4 \cos x \, dx - \int (\sin x)^6 \cos x \, dx$

$= \dfrac{\sin^5 x}{5} - \dfrac{\sin^7 x}{7} + c$

② $\int \cos^5 x \, dx$ $\qquad m=0, \; n=5$

$= \int \cos^4 x \cos x \, dx$

$= \int (\cos^2 x)^2 \cos x \, dx = \int (1 - \sin^2 x)^2 \cos x \, dx$

$= \int (1 - 2\sin^2 x + \sin^4 x) \cos x \, dx$

$= \int \cos x \, dx - 2 \int \sin^2 x \cos x \, dx + \int \sin^4 x \cos x \, dx$

$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + c$

② If the power of Sine is odd $(m = 2k+1)$, Save one Sine and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors interms of cosine

$$\int \sin^{2k+1} x \cos^n x \, dx = \int \sin^{2k} x \sin x \cos x \, dx$$

$$= \int (\sin^2 x)^k \sin x \cos^n x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \sin x \, dx$$

e.g. $\int \sin^3 x \cos^2 x \, dx$

$$= \int \sin^2 x \cos^2 x \sin x \, dx$$

$$= \int (1 - \cos^2 x) \cos^2 x \sin x \, dx$$

$$= \int \cos^2 x \sin x \, dx - \int \cos^4 x \sin x \, dx$$

$$= \int (\cos x)^2 \sin x \, dx - \int (\cos x)^4 \sin x \, dx$$

$$= \frac{-\cos^3 x}{3} + \frac{\cos^5 x}{5} + c$$

③ If both m & n are odd either ① or ② Can be used

(4) If both m & n are even then use

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ & $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

Some times you need to use

$\sin x \cos x = \frac{1}{2} \sin 2x$

e.g $\int \sin^2 x \cos^2 x \, dx$

$= \int \frac{1}{2}(1 - \cos 2x) \frac{1}{2}(1 + \cos 2x) \, dx$

$= \int \frac{1}{4}(1 - \cos 2x + \cos 2x - \cos^2 2x) \, dx$

$= \frac{1}{4}\left[\int (1 - \cos^2 2x) \, dx\right]$

$= \frac{1}{4}\left(\int dx - \int \cos^2 2x \, dx\right)$

$= \frac{1}{4}\left[x - \int \frac{1}{2}(1 + \cos 4x) \, dx\right.$

$= \frac{1}{4}x - \frac{1}{8}\int (1 + \cos 4x) \, dx$

$= \frac{1}{4}x - \frac{1}{8}x - \frac{1}{8}\int \cos 4x \, dx$

$= \frac{1}{8}x - \frac{1}{32} \sin 4x + c$

H.W $\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$ ---- ans

$\frac{1}{4}\left(\frac{3}{2}x - \sin 2x + \frac{1}{8} \sin 4x\right) + c$

(4) products of Sines and Cosines

(a) $\int \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int [\cos(m-n)x - \cos(m+n)x]\,dx$

(b) $\int \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int [\sin(m-n)x + \sin(m+n)x]\,dx$

(c) $\int \cos(mx)\cos(nx)\,dx = \frac{1}{2}\int [\cos(m-n)x + \cos(m+n)x]\,dx$

example: Solve

$$\int \sin 3x \cos 5x\,dx \quad , m=3, n=2$$

by (b)

$$\Rightarrow \int \sin 3x \cos 5x\,dx = \frac{1}{2}\int [\sin(3-5)x + \sin(3+5)x]\,dx$$

$$= \frac{1}{2}\int \sin(-2)x\,dx + \frac{1}{2}\int \sin 8x\,dx$$

$$= \frac{-1}{2}\int \sin 2x\,dx + \frac{1}{2}\int \sin 8x\,dx$$

$$= -\frac{1}{2} * \frac{-1}{2}\cos 2x + \frac{1}{2} * \frac{-1}{8}\cos 8x + C$$

$$= \frac{1}{4}\cos 2x - \frac{1}{16}\cos 8x + C.$$