Derivatives:-

Def:- The derivative of the function $f(x)$ with respect to the varible x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \text{, if the limit exists.}$$

* The derivative of f may denote by the following of the notation

$$f'(x),\ y',\ \frac{df}{dx},\ \frac{dy}{dx},\ \frac{d(f(x))}{dx},\ D_x(y)$$

e.g. $y = \sin x$, find $\frac{dy}{dx}$ by definition?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x \sin h}{h}$$

$$= \lim_{h\to 0} \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{=0} + \lim_{h\to 0} \cos x \cdot \lim_{h\to 0} \frac{\sin h}{h} = 1$$

$$= \cos x$$

* f is called differentiable if $f'(x)$ exists, $\forall x \in D_f$

Differentiation Rules

Let f, g be two differentiable functions and c any real numbers

① $\frac{d}{dx}(c) = 0$ and $\frac{d(x)}{dx} = 1$

② $\frac{d}{dx}(x^n) = n x^{n-1}$, $n > 0$

③ $\frac{d}{dx}(c \cdot f(x)) = c \cdot \frac{df}{dx}$

④ $\frac{d}{dx}(f(x) \mp g(x)) = \frac{df(x)}{dx} \mp \frac{d(g(x))}{dx}$

⑤ $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \frac{df(x)}{dx}$

$$= f \cdot g' + g \cdot f'$$

⑥ $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x) \frac{d}{dx}(f(x)) - f(x) \cdot \frac{d}{dx}(g(x))}{(g(x))^2}$

$$= \frac{g \cdot f' - f \cdot g'}{g^2}$$

Inverse Trig. Functions

① $\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$ ② $\frac{d}{dx}(\cos^{-1}x) = \frac{-1}{\sqrt{1-x^2}}$

③ $\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$ ④ $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$

⑤ $\frac{d}{dx}(\csc^{-1}x) = \frac{-1}{|x|\sqrt{x^2-1}}$ ⑥ $\frac{d}{dx}(\cot^{-1}x) = \frac{-1}{1+x^2}$

Find $\frac{dy}{dx}$ for the following functions:

① $f(x) = \sqrt{x} \Rightarrow f'(x) = \frac{dy}{dx} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

② $f(x) = \ln(e^x + \sin^2 x)$

$$f'(x) = \frac{e^x + \cos x^2 (2x)}{(e^x + \sin x^2)} = \frac{e^x + 2x\cos x^2}{e^x + \sin x^2}$$

$$* \left(\begin{array}{l} \sin^2 x \equiv (\sin x)^2 \\ \sin x^2 \not\equiv (\sin x)^2 \end{array} \right)$$

③ $y = \cos^{-1}(3x^2)$

$$y' = \frac{-6x}{\sqrt{1-(3x^2)^2}} = \frac{-6x}{\sqrt{(1-9x^4)}}$$

④ $y = \ln(\frac{1}{x})$

$\Rightarrow y' = \frac{-1}{x}$ (discuss) , $f'(x) = \frac{1}{\frac{1}{x}} \cdot -x^{-2}$

$= \frac{1}{\frac{1}{x}} \cdot\cdot$ $= \boxed{-\frac{1}{x}}$

Derivatives of Exponential and Logarithm Functions

① $\frac{d}{dx} e^{f(x)} = e^{f(x)} * f'(x)$ / e.g $f(x) = e^{5x} = 5e^{5x}$

② $\frac{d}{dx} a^{f(x)} = a^{f(x)} * \ln a * f'(x)$ / e.g $f(x) = 2^{x^2} \Rightarrow$
$f'(x) = 2^{x^2} \ln 2 . 2x$

③ $\frac{d}{dx}(\ln f(x)) = \frac{1}{f(x)} * f'(x)$ / e.g $f(x) = \ln 2x$
$= \frac{1}{2x}(2) = 1/x$

④ $\frac{d}{dx}(\log_a f(x)) = \frac{(\ln(f(x)))'}{\ln(a)} = \frac{1}{\ln(a)} * \frac{1}{f(x)} \cdot f'(x)$

$f(x) = \log_2 x^2 \Rightarrow f'(x) =$
$\frac{1}{\ln 2} \cdot \frac{1}{x^2} \cdot 2x = \frac{2}{x \ln 2}$

Derivative of Trig. Functions

① $\frac{d}{dx}(\sin x) = \cos x$ ② $\frac{d}{dx}(\cos x) = -\sin x$

③ $\frac{d}{dx}(\sec x) = \sec x \tan x$ ④ $\frac{d}{dx}(\tan x) = \sec^2 x$

⑤ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

⑥ $\frac{d}{dx}(\cot x) = -\csc^2 x$

e.g

(1) $y = x^x \Rightarrow$ take ln of both side

$$\ln y = \ln x^x \Rightarrow \ln y = x\ln x$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \ln x$$

$$= 1 + \ln x$$

$$\Rightarrow \frac{dy}{dx} = y(1+\ln x) \Rightarrow y' = x^x(1+\ln x)$$

(2) $y = (\sin x)^{x^2+3} \Rightarrow \ln y = \ln(\sin x)^{x^2+3}$

$$\ln y = (x^2+3)\ln(\sin x)$$

$$\frac{dy}{dx}\left(\frac{1}{y}\right) = (x^2+3)\frac{\cos x}{\sin x} + \ln\sin x\,(2x)$$

$$\Rightarrow \frac{dy}{dx} = y\left((x^2+3)\frac{\cos x}{\sin x} + 2x\ln\sin x\right)$$

Chain Rule:- If $y=f(u)$ and $u=g(x)$ are differentiable functions ~~then~~

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

(5) $f(x) = 3^x \Rightarrow f'(x) = 3^x\ln(3)\,(1)$

(6) $y = 2^{e^x+\sin x} \Rightarrow y' = 2^{e^x+\sin x}\ln(2)\left(e^x+\cos x\right)$

$$= \left(e^x+\cos x\right)\ln(2)\cdot 2^{e^x+\sin x}$$

(7) $y = \ln(x^2+1) + \ln(x+1)^3$

$$\Rightarrow y' = \frac{2x}{(x^2+1)} + \frac{3\cancel{(x+1)^2}}{(x+1)^{\cancel{3}}} \Rightarrow y' = \frac{2x}{(x^2+1)} + \frac{3}{(x+1)}$$

(8) $y = \ln\sin x^2$

$$\Rightarrow y' = \frac{1}{\sin x^2}(\cos x^2)(2x) = \frac{2x\cos x^2}{\sin x^2} = 2x\cot x^2$$

(9) H.W $y = e^{x^2+1}$, $y = e^{2\ln\sin x}$

$$y = \cos e^{x^2} + \sin\ln x^2$$

(10) If $f(x) = \sqrt{x-2}$, $x \geq 2$, find (y' by def.)

$$f'(x) = \lim_{\Delta x\to 0}\frac{\sqrt{x+\Delta x-2}-\sqrt{x-2}}{\Delta x}\cdot\frac{\sqrt{x+\Delta x-2}+\sqrt{x-2}}{\sqrt{x+\Delta x-2}+\sqrt{x-2}}$$

$$= \lim_{\Delta x\to 0}\frac{\cancel{x}+\cancel{\Delta x}-\cancel{2}-\cancel{x}+\cancel{2}}{\cancel{\Delta x}\left(\sqrt{x+\Delta x-2}+\sqrt{x-2}\right)}$$

$$= \lim_{\Delta x\to 0}\frac{1}{\sqrt{x+\Delta x-2}+\sqrt{x-2}} = \frac{1}{2\sqrt{x-2}}$$

(8)

Higher Derivatives

Given $y = f(x)$, The seconed derivative of f is

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$, Some other notation

$f''(x)$, $D^2 f(x)$, $f^{n}(x)$ for higher than 2

e.g Find $f''(x)$ of $f(x) = x\cos x$

$$f'(x) = x(-\sin x) + \cos x\,(1)$$
$$= -x\sin x + \cos x$$
$$= (-x\cos x - \sin x) - \sin x$$
$$= -(x\cos x + 2\sin x)$$

H.W

(1) If $f(x) = \frac{1}{x}$, then $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$

(2) For $y = \sin x$, find $f^{(99)}$?

(7)

e.g $y = \cos^2 x$ use chain rule to find y'

We have $y = \cos^2 x = (\cos x)^2$

let $u = \cos x$, $y = u^2$

$$\frac{du}{dx} = -\sin x \Rightarrow \frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$
$$= 2u(-\sin x)$$
$$= -2u\sin x = -2\cos x\sin x$$

H.W by chain rule find y' for $y = \cos x^2$

Implicit Differentiation

This method is a special case of the chain rule. using this, we need to differentiate both of the equation with respect to x and then solving the resulting eq. for y'

e.g $x^2 + y^2 = 16 \Rightarrow \frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(16)$

$$2x + 2yy' = 0 \Rightarrow y' = \frac{-x}{y}$$

H.W $x^3 + y^3 = 4xy$, $\cos(x+y) = y^2\sin x$

9

$$y = x^2 \tan\left(\ln \frac{1}{x}\right)$$

$$\Rightarrow y' = x^2 \sec^2\left(\ln \frac{1}{x}\right) \cdot \left(\frac{\frac{-1}{x^2}}{\frac{1}{x}}\right) + \tan\left(\ln \frac{1}{x}\right) 2x$$

$$= -x^2 \sec^2\left(\ln \frac{1}{x}\right)\left(\frac{1}{x}\right) + 2x \tan\left(\ln \frac{1}{x}\right)$$

$$y = \ln \cos^{-1} x$$

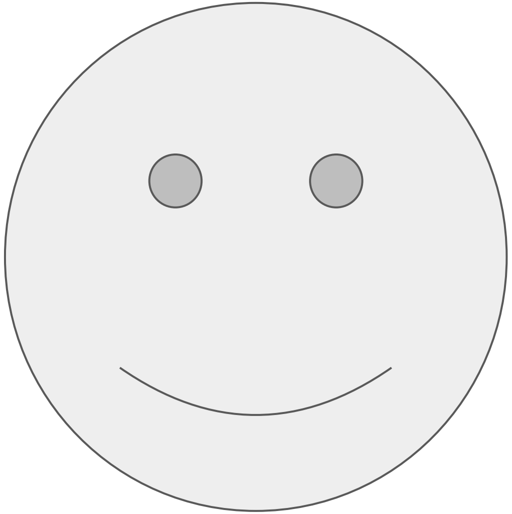
$$\Rightarrow y' = \frac{\frac{-1}{\sqrt{1-x^2}}}{\cos^{-1} x} = \frac{-1}{\cos^{-1} x \sqrt{1-x^2}}$$

$$y = 10^{\tan^{-1}\sqrt{x}} \Rightarrow y' = 10^{\tan^{-1}\sqrt{x}} \ln 10 \, \frac{\frac{1}{2\sqrt{x}}}{1+x}$$

$$\Rightarrow y' = 10^{\tan^{-1}\sqrt{x}} \ln 10 \, \frac{1}{2\sqrt{x}(1+x)}$$

$$y = e^{\cot 2x} \Rightarrow y' = e^{\cot 2x}\left(-\csc^2 2x\right)(2)$$

$$= -2\csc^2(2x)\, e^{\cot(2x)}$$



Thank You