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Random variables

1- Discrete random variables

2- Continuous random variables

(1) we explain first the discrete random variables

The properties of probability distribution for discrete random variables are

1- $p(x) \geq 0$

2- $\sum p(x) = 1$

The distributions followed this prob. are

A- Uniform distributions

Def:- A random variable X has a discrete uniform dist. iff (if and only if) prob. dist is given by:-

$$p(x;k) = \begin{cases} \frac{1}{k}, & x = 1, 2, 3, \ldots, k \\ 0 & \text{other wise} \end{cases}$$

parameter

for example:- $S = \{1, 2, 3, 4, 5, 6\}$

$$p(1) = p(2) = p(3) = \cdots = p(6) = \frac{1}{6}$$

Then $p(x;6) = \begin{cases} \frac{1}{6} & , x = 1, 2, \ldots, 6 \\ 0 & \text{o.w.} \end{cases}$

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Example2. If $K=10$ then $X \sim U(K) \Rightarrow X \sim U(10)$

The prob. is $p(X,10) = \begin{cases} \frac{1}{10} & x = 1,2,3,\dots,10 \\ 0 & o.w \end{cases}$

The Mean of this dist. called Expected of variable $E(X)$ i.e. named μ equal to $\sum x\,p(x)$

Then $E(X) = \mu = mean = \sum x\,p(x)$

$$E(X) = \sum_{x=1}^{K} x \cdot \frac{1}{K} = \frac{\sum_{x=1}^{K} x}{K}$$

$$Var(X) = \sigma_x^2 = E(X-\mu)^2 = \sum (x-\mu)^2 p(x)$$

$$= \sum (x-\mu)^2 \cdot \frac{1}{K} = \frac{\sum_{x=1}^{K} (x-\mu)^2}{K}$$

If we apply the $E(x)$ & $Var(x)$ for the example-1-

imply $E(x) = \mu = mean = \sum_{x=1}^{6} x\,p(x)$

$$= 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6}$$

$$= 3.5$$

or there is another formula to find $E(x)$ & $Var(x)$

that are $E(x) = \frac{K+1}{2}$, $Var(x) = \frac{K^2-1}{12}$

$$var(x) = \sigma^2 = \frac{\sum (x-\mu)^2}{K} = \frac{(1-3.5)^2 + (2-3.5)^2 + \dots + (6-3.5)^2}{6}$$

$$= \frac{35}{12}$$

$$E(x) = \mu = \text{mean} = \frac{K+1}{2} = \frac{6+1}{2} = \frac{7}{2} = 3.5$$

$$V(x) = \sigma^2 = \frac{K^2-1}{12} = \frac{6^2-1}{12} = \frac{36-1}{12} = \frac{35}{12}$$

<u>H.W.</u> Apply the Uniform dist. for this example

Let $y \sim U(10)$ to find

1- The dist. of y

2- The mean of y

3- The variance of y

<u>B</u> - Bernoulli Distributions:

Def:- A random variable X has a Bernoulli dist. if and only if its prob. dist. is given by:-

$$p(x;p) = \begin{cases} p^x (1-p)^{1-x} & x = 0,1 \\ 0 & o.w. \end{cases}$$

p = means prob. of pass

$q = 1-p$ = means prob. of fail

To apply $\sum p(x) = 1$ for this dist. is

$$\sum_{x=0}^{1} p(x;p) = \sum_{x=0}^{1} p^x (1-p)^{1-x} = (1-p) + p = 1-p+p = 1$$

$$E(x) = \text{mean} = \mu = \sum_{x=1}^{n} x p(x) = \sum_{x=0}^{1} x p^x (1-p)^{1-x}$$
$$= p$$

$$\text{var}(x) = \sigma^2_x = E(X-\mu)^2 = E(X^2 - 2X\mu + \mu^2) = EX^2 - 2\mu EX + E\mu^2$$
$$= EX^2 - 2\mu * \mu + E\mu^2 = EX^2 - \mu^2 = \sum_{x=0}^{1} x^2 p^x (1-p)^{1-x} - p^2$$
$$= p - p^2 = p(1-p) = pq, \text{ where } 1-p = q$$

Ex-1- $X \sim \text{Ber}(p)$ if we have $p = \frac{1}{2}$ then

$$X \sim \text{Ber}(\tfrac{1}{2}) \Rightarrow p(x; p) = p^x (1-p)^{1-x} \quad x = 0, 1$$

$$p(x; \tfrac{1}{2}) = (\tfrac{1}{2})^x (1 - \tfrac{1}{2})^{1-x} \quad x = 0, 1$$
$$= (\tfrac{1}{2})^x (\tfrac{1}{2})^{1-x}$$

$$\text{Mean} = E(x) = \mu = p = \tfrac{1}{2}$$

$$\text{variance}(x) = \sigma^2 = \text{var}(x) = pq = (\tfrac{1}{2} \cdot \tfrac{1}{2}) = \tfrac{1}{4}$$

H.w. Find the Ber. dist for $X \sim \text{Ber}(\frac{1}{3})$

1- The dist. of X

2- The mean of X

3- The var(X)

C- Binomial distribution:-

A Bernoulli trial can result in success with prob. (p) and a failure with prob. $q=1-p$. Then the prob. dist. of the binomial random variable X, the number of success in n independent trials is

$$P_r(X;n,p)=\begin{cases} C_x^n\, p^x q^{n-x} & x=0,1,\dots,n \\ 0 & \text{o.w.}\end{cases}$$

Ex:1 Two coins are tossed

$p=\{HH,HT,TH,TT\}$

1- $p(2H)=\frac{1}{4}=p(HH)$

2- $p(2T)=\frac{1}{4}=p(TT)$

3- $p(1H)=p(HT+TH)=\frac{1}{2}$

To prove this example by binomial dist. we see

1- $p(2H)=C_2^2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^0 \quad x=0,1,2,\dots,n \qquad p=\frac{1}{2}$

$\qquad q=1-p=1-\frac{1}{2}=\frac{1}{2}$

$=\frac{2!}{0!\,2!}\left(\frac{1}{4}\right)(1)=\frac{1}{4}$

2- $p(2T)=C_2^2\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^2=\frac{2!}{0!\,2!}(1)\left(\frac{1}{4}\right)=\frac{1}{4}$

3- $p(1H)=C_1^2\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)=2\times\frac{1}{4}=\frac{1}{2}$

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Ex-2- Find the prob. of getting five heads in (12) slips of a blanced coin, using binomial dist.

Solⁿ: $pr(x; n, p) = C_x^n p^x (1-p)^{n-x}$ $x = 0, 1, --, 5$

$n = 12$, $x = 5$, $p = \frac{1}{2} = 0.5$ $q = 1-p = 1-\frac{1}{2} = \frac{1}{2}$

$pr(5; 12, \frac{1}{2}) = C_5^{12} (\frac{1}{2})^5 (1-p)^{12-5}$

$= C_5^{12} (\frac{1}{2})^5 (\frac{1}{2})^7 = \frac{12!}{5!\,7!} (\frac{1}{2})^5 (\frac{1}{2})^7$

$= 0.19$

if $x = 0$ i.e. no heads then,

$p(\text{No heads}) = pr(0, 12, \frac{1}{2}) = C_0^{12} (\frac{1}{2})^0 (\frac{1}{2})^{12}$

$1 \times 1 \times (\frac{1}{2})^{12} = (\frac{1}{2})^{12} = \frac{1}{4096} = 0.0002$

H.w. ① if $x \sim bin(20, \frac{1}{3})$ find the prob. dist. of binomial for x

Therom. The mean or $E(x)$ or μ for Binomial is

$E(x) = \mu = mean = np$

and $var(x) = \sigma^2 x = npq = np(1-p)$

H.w. ② Find the $E(x)$ and $var(x)$ if the $y \sim bin(10, \frac{1}{4})$ and find the dist. of y

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D- poisson distribution:

Def:- A random variable X has a poisson distribution if and only if prob. dist. is given by:-

$$Pr(x;\lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & x=0,1,2,\ldots \\ 0 & o\text{-}w. \end{cases}$$

The mean $= E(x) = \mu = \lambda$

$\text{varian}(x) = \sigma_x^2 = \mu = \lambda$

EX:- The number of cars aboundoned weekly on a certain highway has apoisson dist. with $\lambda = 2.2$ in given week, what is the prob. that.

1- No, cars aboundoned?

2- exactly one car is aboundoned?

3- at most " " " " ?

4- at least " " " " ?

sol

$$p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\ldots$$

1- $p(X=0, \lambda=2.2) = \dfrac{e^{-2.2}\,2.2^{0}}{0!} = e^{-2.2} = 0.1108$

2- $p(X=1, \lambda=2.2) = \dfrac{e^{-2.2}\,2.2^{1}}{1!} = (2.2)(e^{-2.2}) = 0.2438$

3- $p(X\leq 1, \lambda=2.2) = p(X=1)+p(X=0)$

$= 0.2438 + 0.1108 = 0.3546$

4- $p(X\geq 1) = p(X=1)+p(X=2)+\ldots$

$= 1-p(X<1) = 1-p(X=0) = 1-0.1108 = 0.8892$

$Mean = E(X) = \mu = \lambda = 2.2$

$Var(X) = \sigma_x^2 = \mu = \lambda = 2.2$

H.w. if $X \sim Poi(5)$, Find the

1- Dist. of X
2- The mean of X
3- The variance of X

E- Geometric distribution:

Def:- A random variable X has a geometric dist. and it is reffered to as a geometric random variable if and only if it is prob. dist. is given by

$$G(X; \theta) = \theta(1-\theta)^{X-1} \quad X = 1, 2, ---$$

Ex:- In a certain manufacturing process it is known that in the average for every 100 items is defective what is the prob. that the fifth item expected is the first defective item found.

sol:- Using by geometric dist. with $X=5$

$$g(5, 0.01) = (0.01)(0.99)^{5-1} \qquad \theta = \frac{1}{100} = 0.01$$
$$= 0.0096$$

Theorem The mean and variance of the random variables of the geometric dist. are.

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$\mu = \text{mean} = E(x) = \frac{1}{p}$

$var(x) = \sigma_x^2 = \frac{1-p}{p^2} = \frac{q}{p^2}$

or $f(x;p) = p(1-p)^x \quad x = 0, 1, 2, ---$

$= p q^x \quad \text{where } q = 1-p$

Then $x \sim G(q)$

or $x \sim G(p)$

Ex: $x \sim G(0.4)$, find the dist of x

= mean of x

= var(x) of x

1- $f(x;p) = p(1-p)^{x-1} \quad x = 1, 2, ---$

$= 0.4(0.6)^{x-1} \quad x = 1, 2, ---$

2- Mean $= E(x) = \mu = \frac{1}{p} = \frac{1}{0.4} = 2.5$

3- variance $= \sigma_x^2 = \frac{q}{p^2} = \frac{0.6}{(0.4)^2} = 3.75$

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