Introduction of probability
Techniques of Counting

Factorial notation

It is based on the fact that n different object can be arranged in the form of a sequence in n factorial number of ways which is known as permutation of a set of objects. This concept was discovered by Indian scholars in early 12th century. The product of one to n is called as factorial of n and denoted as $n!$ (Where n is a whole number).
The operation of factorial is utilized in many different branches of mathematics, most notably in algebra, calculus, mathematical analysis and even in combinatory.
Factorials are used to find the different number of ways in order to arrange a set of objects in a sequence. Just for an example, if we have **4** pens on the table and we are to arrange them in a sequence. Then, these can be arranged in factorial **4** number of different ways.

What is meant by the Factorial?

In Mathematics, the factorial of a number (n) is defined as a number which is the product of all the positive numbers which is equal to n. The factorial is represented using the exclamation mark "!". The factorial is extensively used in permutations, combinations and in probability problems. In other words, a factorial is a function that multiplies the number by every number below it.

For example, the factorial of 5 is 120.

Here n=5,

Hence, the factorial of 5 is $5 \times 4 \times 3 \times 2 \times 1 = 120$

Definition and Symbol

In math's, the factorial of a *positive integer is defined as the product of all the positive integers which are less than or equal to that given number*. In other words, the product of some number and all the positive numbers less than that is known as the factorial. The factorial of n indicates the multiplication of series of positive integers starting from n and descending till one. According to one more definition
- The factorial of a positive number is represented by the product of all whole numbers smaller than the given number, except zero. **The factorial is denoted by "!".**

For Example: the factorial of a positive integer n is represented by the symbol **n!** (This symbol introduced in **1808** by mathematician **Christian Kramp**). The factorial notation is read as "***n* factorial**". Such as 5! is pronounced as "5 factorial".

The formula for the calculating of factorial of a positive integer *n* is illustrated by

$$n! = n. (n - 1). (n - 2)... 3. 2. 1$$

$$n! = n (n-1)!$$

$$n! / n = (n-1)!$$

There is a special case of factorial of zero which is defined as equal to 1; **i.e. 0! = 1**
This is because of the interpretation that there must be exactly one way for arranging zero things. This means that there is one single permutation for zero objects, known empty or null set.

For example:

$$4! = 4 (4-1)(4- 2)(4- 3)= 4. 3. 2. 1= 24$$

below, there is a list of few factorials. Let us have a look:

0! = 1
1! = 1
2! = 2. 1 = 2
3! = 3. 2. 1 = 6
4! = 4. 3. 2. 1 = 24
5! = 5. 4. 3. 2. 1 = 120
6! = 6. 5. 4. 3. 2. 1 = 720
7! = 7. 6. 5. 4. 3. 2. 1 = 5040

Some simple examples based on factorials are illustrated below:

Example 1: Find the value of 10!
Sol. 10! = 10 (10 - 1)(10 - 2)(10 - 3)(10 - 4)(10 - 5)(10 - 6)(10 - 7)(10 - 8)(10 - 9)
= 10. 9. 8. 7. 6. 5. 4. 3. 2. 1 = 362880

Example 2: Solve 9!/ 3! 4!
Sol. 9!/ 3! 4! = 9.8.7.6.5.4!/3! 4!= 9.8.7.6.5/3!= 9.8.7.6.5/6= 9. 8. 7. 5= 2520

Example 3: 5!+0!+3!/ 3
=5*4*3*2*1+1+3*2*1/3=120+1+2=123

Example 4: Solve $(k+1)!/ (k-1)!$
Sol. $(k+1)! / (k-1)!$
$= (k+1)(k).(k-1)(k-2)...3.2.1/ (k-1)(k-2)...3.2.1$
$= (k+1)(k)$
$= k^2 + k$

Example 5: Solve 10!/ 5! 2!

Sol. Given, 10! / 5! 2!

10! / 5! 2! = 10∗9∗8∗7∗6∗5! / 5!* 2∗1 = 10 * 9 * 8 * 7 * 3= 15120

Example 6: Simplify the following expressions:

(n + 2)! / n!

Solution. (n+2)!/ n!= [1* 2 *…* n * (n+1) *(n+2)]/ [1* 2* …* n]

= (n+1) (n+2)

Example 6: Write in terms of factorials

5*10*15*20*25

= 1*5 *2*5 *3*5 *5*4 *5*5

= 5^5*5!

Permutation

Each of the different arrangement which can be made by taking some or all of a number of objects is called permutation.

Permutation of *n* different objects

The number of arranging of n objects taking all at a time, denoted by ^nP_n, is given by ^nP_n = n!

The number of an arrangement of n objects taken r at a time, where $0 < r \leq n$, denoted by nP_r is given by

$^nP_r = n!/(n-r)!$

Properties of Permutation

(i) $^nP_n = n(n-1)(n-2)\ldots 3\times 2\times 1 = n!$

(ii) $^nP_0 = \dfrac{n!}{n!} = 1$

(iii) $^nP_1 = n$

(iv) $^nP_{n-1} = n!$

(v) $^nP_r = n \, . \, ^{n-1}P_{r-1} = n(n-1)\,^{n-2}P_{r-2}$

(vi) $^{n-1}P_r + r \, . \, ^{n-1}P_{r-1} = {}^nP_r$

(vii) $\dfrac{^nP_r}{^nP_{r-1}} = n - r + 1$

Important Results on Permutation

The number of permutation of *n* things taken *r* at a time, when repetition of object is allowed is *nr*.

The number of permutation of n objects of which p_1 are of one kind, p_2 are of second kind,… p_k are of k^{th} kind such that $p_1 + p_2 + p_3 + \ldots + p_k = n$ is n!p1!$p_{2!}p_{3!}$…..$p_{k!}$

Number of permutation of n different objects taken r at a time,

When a particular object is to be included in each arrangement is r. ${}^{n-1}P_{r-1}$

When a particular object is always excluded, then number of arrangements = ${}^{n-1}P_r$.

Number of permutations of n different objects taken all at a time when m specified objects always come together is m! (n – m + 1)!.

Number of permutation of n different objects taken all at a time when m specified objects never come together is n! – m! (n – m + 1)!.

Combinations

Properties of Combinations

(i) ${}^nC_0 = {}^nC_n = 1$

(ii) ${}^nC_1 = {}^nC_{n-1} = n$

(iii) ${}^nC_r = \dfrac{{}^nP_r}{r!}$

(iv) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(v) ${}^nC_r = {}^nC_{n-r}$

(vi) $r\ {}^nC_{r-1} = (n - r + 1)\ {}^nC_{r-1}$

Concepts of the permutations and combinations:-

The permutation and combination takes place in different type. The **permutation** is an arrangement of object in different ways they can be ordered i.e. first, second, third, etc. If you desire to choose some objects from a larger number of objects, the way you place the chosen object is also important. And the **combination** is the selection of the objects. *<u>It does not consider</u>* the order in which objects were selected or placed, just which objects were selected.

The formulas for permutation and combination are defined as follows:

The number of permutations of *n* objects taken *r* at a time is defined to be

P (n, r)= *n*!/ (*n*−*r*)! and r<=n for example 5p3=5!/(5-3)!=5*4*3*2!/2!=60

The number of permutations of *n* objects taken *r* at a time when repetition is allowed= $\mathbf{n^r}$.
Note: - where n= r **==>** npr= n! ==> npn= n! for example 5p5= 5!/(5-5)!= 5!/0!= 5!/1= 5!
where r= 0 ==> np0= 1, for example 5p0= 5!/(5-0)!= 5!/5!= 1

Ex1:- In how many ways can 10 people be seated on bench if only 4 seats are an available. 10p4=10! /(10-4)!=10*9*8*7*6!/6!=5040.

Ex2. By how many ways can arrangement letters words "Mathematics"

N=11 ==> M=2, a=2, t=2, h=1, e=1, i=1, c=1, s=1

$$p_r^n = \frac{n!}{n1!n2!.....nm!} = \frac{11!}{2!2!2!} = 4989600$$

Ex3. Find the number *n* of seven letters from the word "BENZENE" that can be formed

N=7 ==> B=1, E=3, N=2, Z=1

$$p_r^n = \frac{n!}{n1!n2!.....nm!} = \frac{7!}{1!3!2!1!} = 420$$

Permutations

There are two types of permutations as follows:

- Permutation with repetition (replacement).
- Permutation without repetition (without replacement).

Permutation with Repetition:

When we have *n* different objects, then we have *n* choices each time. And if we are in a position to choose object *r* from *n* objects, the permutations are

n * n * n…..(r times) = n^r i.e. P (n, r) = n^r

For example: In how many ways can a man put 4 balls in 3 bags?

Sol. First ball can put in 3 ways.
Second ball can put in 3 ways.
Third ball can put in 3 ways.
Fourth ball can put in 3 ways.
So the man can put the 4 ball in 3 * 3 * 3 * 3 = 3^4 = **81** ways.

Permutation without Repetition:

In permutation without repetition, we have to reduce the number of available choices each time. When we have *n* different objects, then we have to reduce 1 from the previous term for each time. This is like n * (n - 1) * (n -2)…

P (n, r) = *n*! / (*n*−*r*)! and r ≤ n

Ex.: - Three cards are chose *n* in succession from card play with 52. Find the number of ways his can be done

a- with replacement.
b- without replacement.

Sol.

a- n^r= $(52)^3$=52*52*52= *140608*

b- 52*51*50=132600

or

b- p_r^n =52!/ (52-3)! =132600

Combination

The number of combinations of *n* objects taken *r* at a time and if we are in a position to choose *r* objects from *n* objects, the combination is defined to be

$$C\ (n, r)=n!/\ (n-r)!\ r!$$

Note: - **if r=n** ==> nCr=1

if r=0 ==> nC0=1

and the equation relating the permutation and combination formulas is,

$$C\ (n, r) = P\ (n, r)\ r!$$

Permutation of *r* things out of *n*, involves two activities.

- *r* objects are selected from a set of *n* (Combination C(n, r)).
- The selected *r* objects are arranged in the given order (permutation P(r, r) = r!)

For each selection, there is r! arrangements. Hence, C (n, r) selections yield totally C (n, r) x r! permutations which is shown by the equation. While deciding which formula to use in problems, if the scenario described involves only selections and the order is not required, then go to the combination formula.

On the other hand, where ever order is perceived settle of permutations.

- Permutations - Means arrangement or ordering is required
- Combinations - Means Pure selection is involved

Combination

Also there are two types of combination as follows:

- Combination with Repetition.
- Combination without Repetition.

Combination with Repetition:

We need to do is alter our permutations formula to reduce it by how many ways the objects could be in order but the order is not important here.

$$C\ (n, r) = n!\ /\ r!\ (n-r)!$$

Combination without Repetition:

When we have *n* different objects and to select *r* objects without repetition we have a formula

$$C\ (n, r) = n+r-1/\ r!(n-r)!$$

Note: - If r=n then nCr=1 and r$\leq$n for examples 3C3=3! / 3! (3-3)! =1
If r=0 then nC0=1 for examples 3C0=3! / 0! (3-0)! =1

Ex. 1: Find 7C4- 6C5- 4C4

Sol.

7! /4!(7-4)! - 6! /5!(6-5)!- 1

= 7! /4!*3! - 6! /5!*1!-1=7*6*5*4! /4!*3!-6*5! /5!*1-1=35-6-1=28

Ex. 2: Write all the combination of four balls taken one at a time.

Sol.

Here n = 4 and r = 1= 4! / 1! (4−1)! = 4∗3∗2∗1/ 1∗3! = 4∗3∗2∗1/ 1∗3∗2∗1= 4

Ex. 3: Solve C (5, 2) * C (4, 3)

Sol.

C (5, 2)*C (4, 3) = (5! / 2! 3!)*(4! /3! 1!)= (5∗4∗3! /2∗3!) *(4∗3! /3!)= 10*4 = 40.

Ex. 4: A farmer buys 3 Cows, 2 Pigs and 4 Hens from a person, how has 6 Cows, 5 Pigs and 8 Hens. How many choices the farmer choice these?

Sol.

$C_3^6 . C_2^5 . C_4^8$ = (6!/3!*3!) (5!/2!*3!) (8!/4!*4!)

= (6*5*4*3! /3!*3!)(5*4*3! /3!*3!)(8*7*6*5*4! /4!*4!)

= 20*10*70=14000

Ex. 5: A student is to answer 8 out of 10 questions in exam

a. Find the number *n* of ways that the student can choice the eight questions

b. Find the number if the must answer the first three questions.

Sol.

C_8^{10} = 10!/8!*2!= 10*9*8!/ 8!*2!= 45

C_5^7 =7!/5!*2!= 7*6*5!/5!*2!= 21

Permutation and combinations word problems

Let us see how to solve word problems of permutation and combination.

Question1: At providence high school, students are to elect 6 representatives for Student's council. There were 15 names on the ballot paper.

1. How many different 6 member councils can be formed?

2. If the elected members of the council are to take up different responsibilities, then how many types of council can be formed?
3. There are 8 boys and 7 girls contesting the election, how many committees of 3 boys and 3 girls can be formed?

Answer

a) This is a case of pure selection as ordering is not required here. 6 members are to be selected from 15 candidates. This can be done in **C (15, 6) ways**

$$C(n, r) = n!/ r!\,(n-r)!$$

$$= 15! / 9!\,6! = 15*14*13*12*11*10*9! / 9! *6*5*4*3*2*1 = 5005$$

The student representative council can be formed in 5005 ways.
b) The different responsibilities assigned to the council members brings in order, hence this counting is a candidate for applying permutation formula.
6 member council can be formed from 15 candidates with different responsibilities in **P (15, 6) ways.**

$$P(n, r) = n!/ (n-r)!$$

$$P(15, 6) = 15!/ 9! = 3{,}603{,}600$$

The 6 member council with members holding different responsibilities can be formed from the 15 candidates in 3,603,600 ways.
c) Here order is not required. But three questions are to answer in sequence as follows.

I. How many ways 3 boys can be chosen from 8 male candidates?

II. How many ways 3 girls can be chosen from 7 female candidates?

III. How many ways 3 boys and 3 girls chosen together?

Now.

3 boys from 8 male candidates can be chosen in **C(8, 3) ways = 56 ways**
3 girls from 7 female candidates can be chosen in **C(7, 3) ways = 35 ways**
==> A council consisting of 3 boys and 3 girls can be formed in = 56 x 35 = 1960 ways.

Question2: How many 2 digit numbers can be formed 7, 5, 4 and 2?

$$p_2^4 = \frac{4!}{(4-2)!} = \frac{4*3*2*1}{2*1} = 12$$

i. e. **57, 75, 74, 47, 54, 45, 24, 42, 27, 72, 52, 25**