Limits

Def:- let $y=f(x)$ be any function, then we say that, when x is closet to a, then $f(x)$ is closet to L, we can write the above statements as:-

$$\lim_{x \to a} f(x) = L$$

e.g

$$\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} x+3 = 6$$

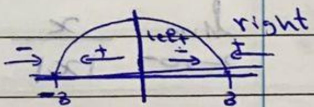
Note that when $x \to 3$ doesnot mean that $x=3$.

* we write $\lim_{x \to a^-} f(x) = L$ & say that the left hand side limit of $f(x)$ as $x \to a$ is equal to L & $\lim_{x \to a^+} f(x) = L$, right hand side.

* $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$

There are functions that might have left ~~hand~~ hand side or right hand side limit or may be have them both but not equal.

e.g



$f(x) = \sqrt{9-x^2}$

$\lim_{x \to -3^+} \sqrt{9-x^2} = 0$, $\lim_{x \to -3^-} \sqrt{9-x^2}$ undefined

$\lim_{x \to 3^-} \sqrt{9-x^2} = 0$, $\lim_{x \to 3^+} \sqrt{9-x^2} =$ undefined

Laws of limits:-

Let c be a constant and the limits $\lim_{x\to a} f(x)$ & $\lim_{x\to a} g(x)$, exist then

1. $\lim_{x\to a}(f(x) \mp g(x)) = \lim_{x\to a} f(x) \mp \lim_{x\to a} g(x)$

2. $\lim_{x\to a}(c.f(x)) = c.\lim_{x\to a} f(x)$

3. $\lim_{x\to a}(f(x).g(x)) = \lim_{x\to a} f(x).\lim_{x\to a} g(x)$

4. $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$, if $\lim_{x\to a} g(x) \neq 0$

5. $\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n$, $n>0$ integer

6. $\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, $n \in \mathbb{Z}^+$

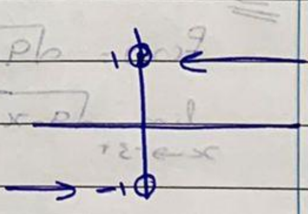
e.g.

$f(x) = \dfrac{x}{|x|}$

$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

$\lim_{x\to 0^-}\dfrac{x}{|x|} = \lim_{x\to 0^-}\dfrac{x}{-x} = -1$

$\lim_{x\to 0^+}\dfrac{x}{x} = 1$



e.g ① $\lim_{x \to 2} x^2 + 3x - 2 = 4 + 6 - 2 = 8$

② $\lim_{x \to -1} \frac{x^2+4}{2-x} = \frac{1+4}{2+1} = \frac{5}{3}$

* If $f(x) = k$, where k is a constant function

$\Rightarrow \lim_{x \to x_0} f(x) = \lim_{x \to x_0} k = k$

e.g $\lim_{x \to 3} 4 = 4$

e.g $\lim_{x \to -2} (2x^4 - 7x^3 + 1)$

$= \lim_{x \to -2} 2x^4 - \lim_{x \to -2} 7x^3 + \lim_{x \to -2} 1$

$= 2(-2)^4 - 7(-2)^3 + 1 = (\quad)$

e.g $\lim_{x \to 3} \sqrt{x^3 - 2}$

$= \sqrt{\lim_{x \to 3} (x^3 - 2)} = \sqrt{3^3 - 2} = \sqrt{27 - 2} = \sqrt{25} = 5$

e.g $\lim_{x \to -2} 6^{1-x^2} = 6^{\lim_{x \to -2}(1-x^2)} = 6^{1-(-2)^2} = 6^{1-4} = 6^{-3} = \frac{1}{6^3}$

$= \frac{1}{216}$

e.g $\lim_{x \to 1} \log_5 (x^6 + 5x - 1)$

$= \log_5 \lim_{x \to 1} (x^6 + 5x - 1) = \log_5 (1^6 + 5(1) - 1)$

$= \log_5 5 = 1$

Essential limits

① $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$ ② $\lim\limits_{x\to 0} (1+x)^{\frac{1}{x}} = 0$

③ $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{x} = 1$ ④ $\lim\limits_{x\to 0} \dfrac{e^{x}-1}{x} = 1$

⑤ $\lim\limits_{x\to 0} \dfrac{\log_a(1+x)}{x} = \lim\limits_{x\to 0} \log_a \dfrac{(1+x)}{x} = \dfrac{1}{\ln a}$

⑥ $\lim\limits_{x\to 0} \dfrac{a^{x}-1}{x} = \ln a$ ⑦ $\lim\limits_{x\to 0} \dfrac{(1+x)^{n}-1}{x} = n$

⑧ $\lim\limits_{x\to 0} \dfrac{\cos x-1}{x} = 0$ ⑨ $\lim\limits_{x\to 0} \dfrac{\sin^{-1} x}{x} = \dfrac{1}{\lim\limits_{x\to 0} \dfrac{\sin x}{x}} = \dfrac{1}{1} = 1$

e.g ① $\lim\limits_{x\to 0} \dfrac{\sin(-x)}{x} = \lim\limits_{x\to 0} \dfrac{-\sin x}{x} = -\lim\limits_{x\to 0} \dfrac{\sin x}{x} = -1$

② $\lim\limits_{x\to 0} \dfrac{\tan x}{x} = \lim\limits_{x\to 0} \dfrac{\sin x}{x\cos x} = \lim\limits_{x\to 0} \left(\dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}\right)$

$= \lim\limits_{x\to 0} \dfrac{\sin x}{x} \cdot \lim\limits_{x\to 0} \dfrac{1}{\cos x} = 1 \cdot 1 = 1$

e.g $\lim\limits_{x\to 1} \dfrac{\sin x - \sin 1}{x-1}$, let $y = x-1 \Rightarrow x = y+1$

$y \to 0$ as $x \to 1$

$\lim\limits_{x\to 1} \dfrac{\sin x - \sin 1}{x-1} = \lim\limits_{y\to 0} \dfrac{\sin(1+y) - \sin 1}{y}$

$= \lim\limits_{y\to 0} \dfrac{\sin 1 \cos y + \cos 1 \sin y - \sin 1}{y} = \lim\limits_{y\to 0} \dfrac{\sin 1(\cos y - 1)}{y} + \lim\limits_{y\to 0} \dfrac{\cos 1 \sin y}{y}$

$= \sin 1 \underbrace{\lim\limits_{y\to 0} \dfrac{(\cos y - 1)}{y}}_{=0} + \cos(1) \underbrace{\lim\limits_{y\to 0} \dfrac{\sin y}{y}}_{1} = \cos 1$

(2) $\lim_{x \to 3} \frac{3^x - x^3}{x-3}$, let $y = x-3$, $x = y+3$

$y \to 0$ as $x \to 3$

$$\lim_{x \to 3} \frac{3^x - x^3}{x-3} = \lim_{y \to 0} \frac{3^{(y+3)} - (y+3)^3}{y}$$

$$= \lim_{y \to 0} \frac{3^y 3^3 - 3^3 (1+y/3)^3}{y}$$

$$= 3^3 \lim_{y \to 0} \frac{3^y - (1+\frac{y}{3})^3 + 1 - 1}{y}$$

$$= 3^3 \lim_{y \to 0} \frac{(3^y - 1) - [(1+\frac{y}{3})^3 - 1]}{y}$$

$$= 3^3 \left[\lim_{y \to 0} \frac{3^y - 1}{y} - \lim_{y \to 0} \frac{(1+\frac{y}{3})^3 - 1}{y} \right]$$

$$= 3^3 [\ln 3 - \frac{1}{3} \cdot 3] = 3^3 [\ln 3 - 1]$$