~~Chapter~~

Chapter One

Symbols:

(A, B, C, ... [sets] , a, b, c, ... , [elements or points]

belong to — subset

$\in$, $\notin$, $\subset$, [$\subseteq$] , $\not\subset$, $\cap$, $\cup$, $\setminus$ except , $\forall$ for all ,

$\exists$, $\Rightarrow$ implies that , $\Leftrightarrow$ if and only if , ...)

Numbers Systems

1/ The natural numbers ($\mathbb{N}$), namely 1, 2, 3, 4, ...

2/ The integer numbers ($\mathbb{Z}$) namely 0, ∓ 1, ∓ 2, ∓ 3, ...

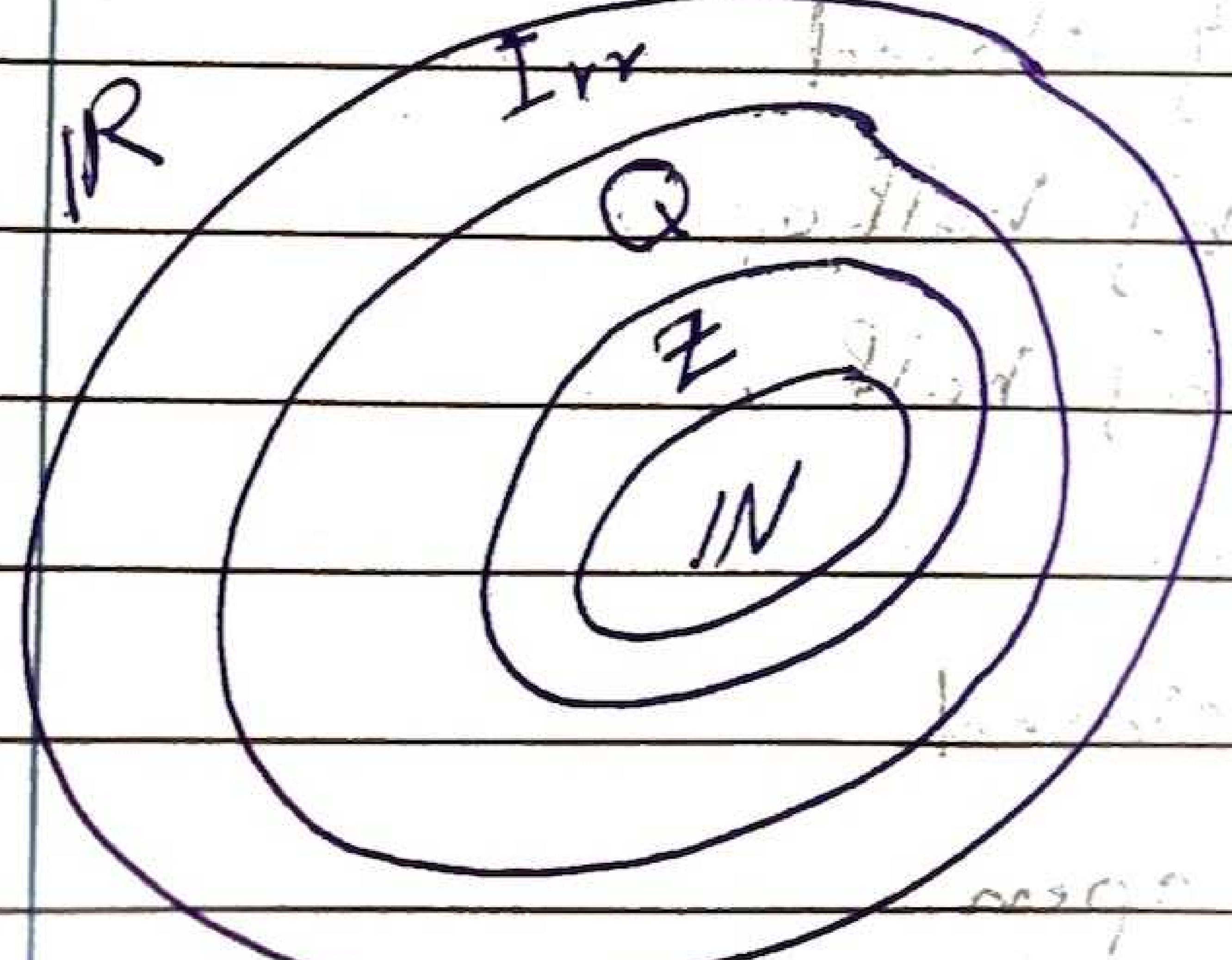
3/ The rational numbers ($\mathbb{Q}$) : the numbers that can be expressed in the form of a fraction m/n , Where m, n are integers and $n \neq 0$ ($\frac{1}{3}$, $\frac{-4}{9}$, $\frac{200}{13}$ ...

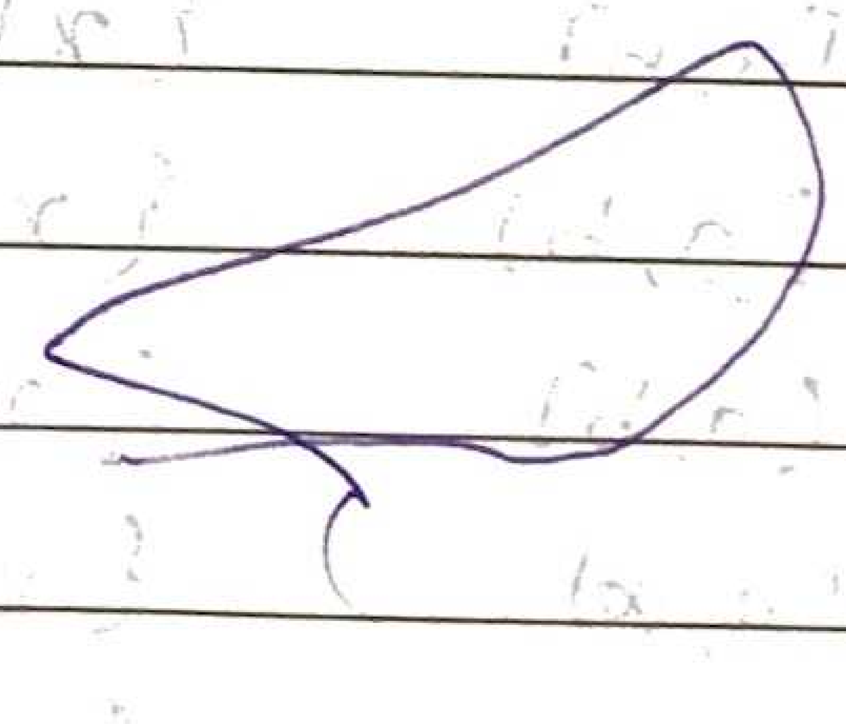
4/ The irrational numbers ($Irr = Q^c$) : Real numbers that are not rational number (π , $\sqrt{2}$, $\sqrt[3]{5}$, $\log_{10} 3$

5/ Real number

$$\mathbb{N} \subseteq \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

$$Q^c \subseteq \mathbb{R}$$





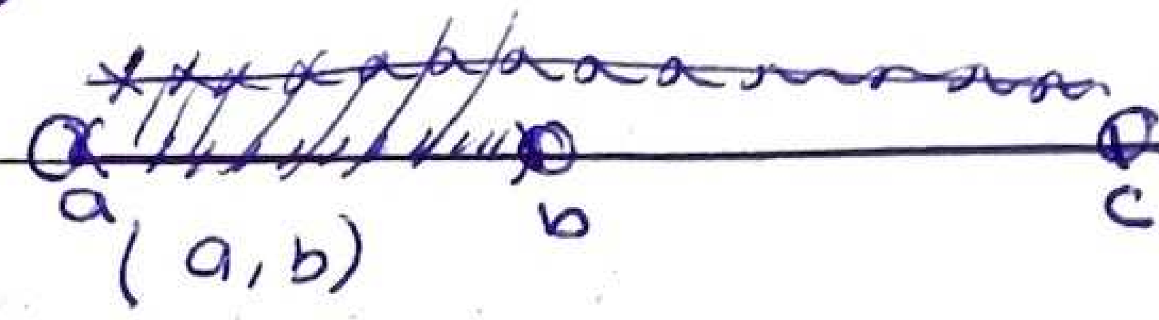
Union and Intersection

The union of two sets is everything that is in either of the set (or both)

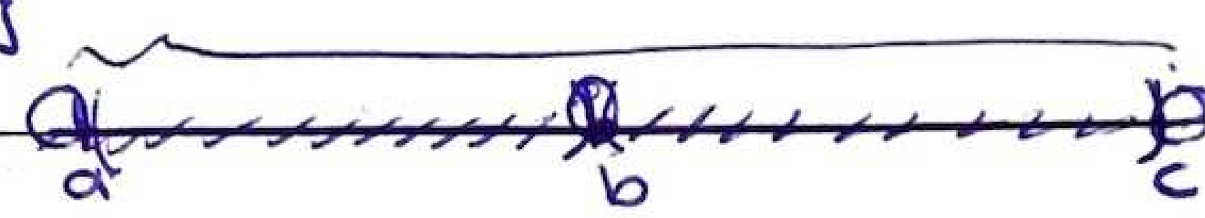
The intersection of two sets is every thing that the sets have in common.

If $a, b, c \in \mathbb{R}$ s.t. $a < b < c$ then

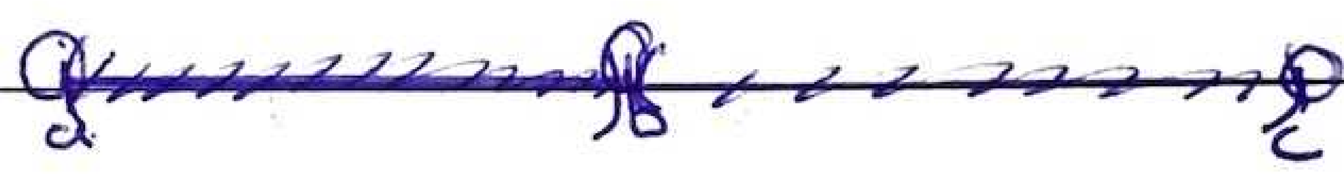
① $(a,b) \cap (a,c) = (a,b)$



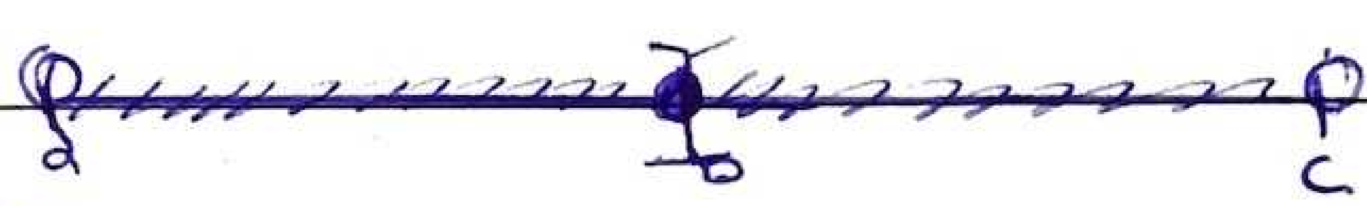
② $(a,b) \cup (b,c) = (a,c) \setminus \{b\}$



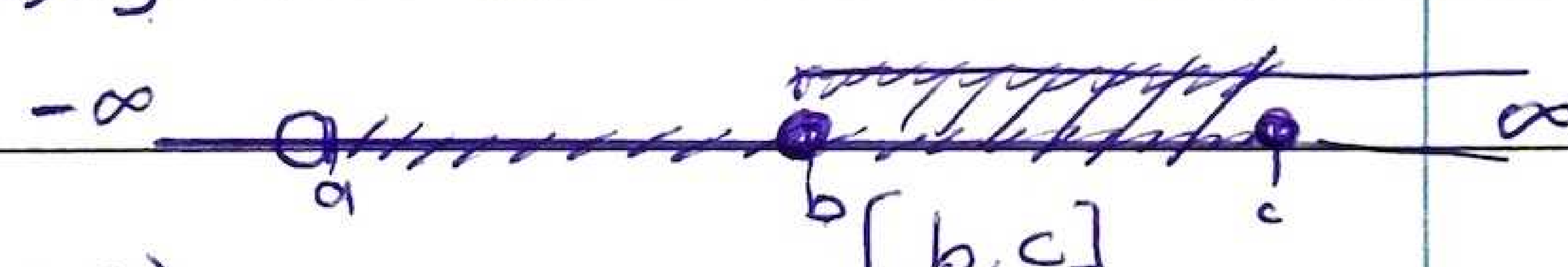
③ $(a,b) \cap (b,c) = \emptyset$



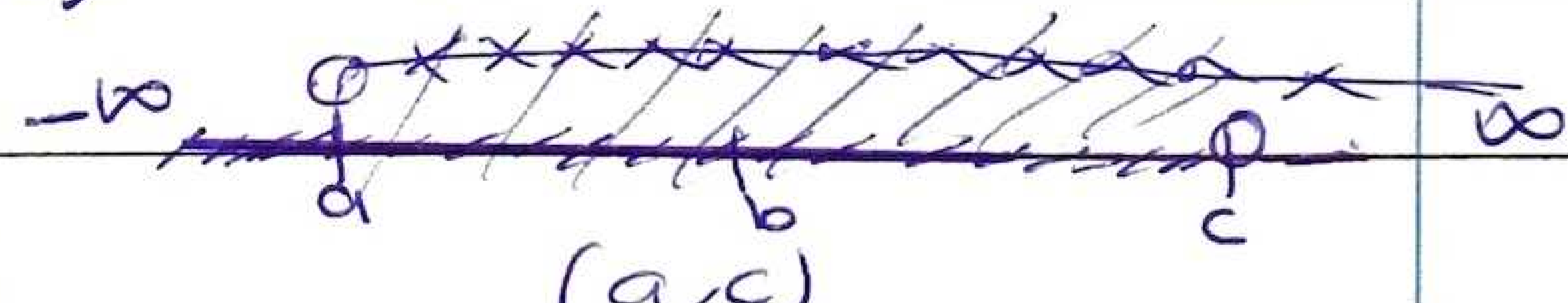
④ $(a,b] \cap [b,c) = \{b\}$



⑤ $(a,c] \cap [b,\infty) = [b,c]$

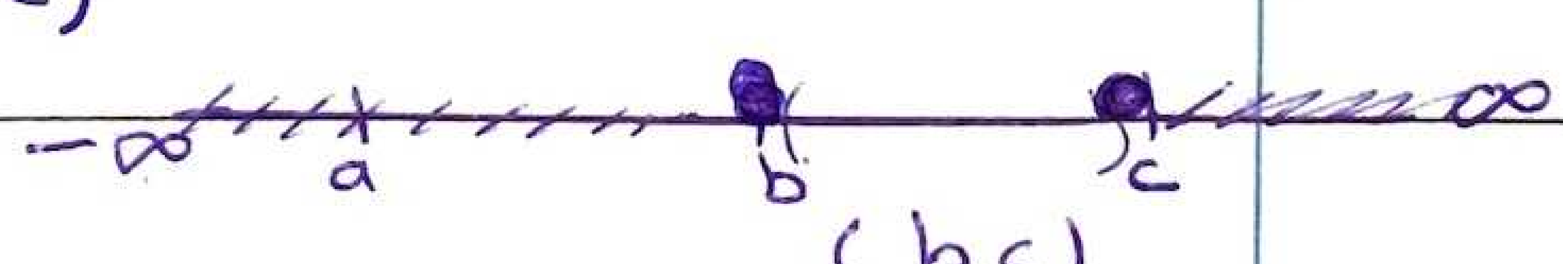


⑥ $(-\infty,c) \cap (a,\infty) = (a,c)$



⑦ $(-\infty,b) \cup (c,\infty) = \mathbb{R} \setminus [b,c]$

⑧ $(-\infty,b] \cup [c,\infty) = \mathbb{R} \setminus (b,c)$



⑨ $(-\infty,a) \cup [a,\infty) = \mathbb{R}$

notation

finite :-

	(a,b)	$\{x \mid a < x < b\}$	open
	$[a,b]$	$\{x \mid a \leq x \leq b\}$	closed
	$[a,b)$	$\{x \mid a \leq x < b\}$	halfopen
	$(a,b]$	$\{x \mid a < x \leq b\}$	halfopen

infinite

	(a,∞)	$\{x \mid x > a\}$	open
	$[a,\infty)$	$\{x \mid x \geq a\}$	closed
	$(-\infty,b)$	$\{x \mid x < b\}$	open
	$(-\infty,b]$	$\{x \mid x \leq b\}$	closed
	$(-\infty,\infty)$	$\mathbb{R}$ (set of all real numbers)	

both closed & open

~~$(-\infty, b]$ $\{x \mid x \leq b\}$ Closed~~

~~$(-\infty, \infty)$ $\mathbb{R}$ (set of all real numbers) Both open & closed~~

Inequalities:- We are dealing with $<$, $>$, $\leq$ and $\geq$.

Properties of Inequalities

Let x, y and $z \in \mathbb{R}$ s.t

① Transitive property:-

If $x \leq y$ and $y \leq z$ then $x \leq z$

If $x \geq y$ and $y \geq z$, Then $x \geq z$

② Addition and subtraction:-

If $x \leq y$, then $x - c \leq y - c$

If $x \leq y$, then $x + c \leq y + c$

If $x \geq y$, then $x - c \geq y - c$ and

If $x \geq y$, then $x + c \geq y + c$

③ Multiplication and Division

If $x \leq y$ and c is positive, then $xc \leq yc$

If $x \leq y$ and c is negative, then $xc \geq yc$

④ Addition Inverse

If $x \leq y$, then $-x \geq -y$

If $x \geq y$, then $-x \leq -y$

5. Multiplicative Inverse

If $x < y$, then $\frac{1}{x} > \frac{1}{y}$

If $x > y$, then $\frac{1}{x} < \frac{1}{y}$

e.g. Find the solution set of the following inequalities:

1. $6x - 4 > 5$

sol $6x - 4 + 4 > 5 + 4$

$6x > 9 \quad \div 6$

$x > \frac{9}{6} \Rightarrow x > \frac{3}{2}$

The solution set (S.S) is $(\frac{3}{2}, \infty) = \{x \in \mathbb{R} : x > \frac{3}{2}\}$

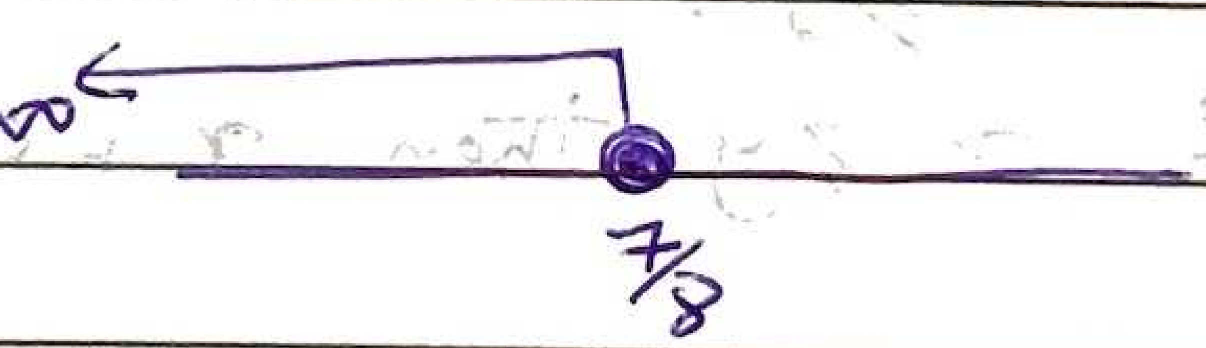
2. $2x - 3 \leq 4 - 6x$

sol $2x - 3 \leq 4 - 6x$ (x's in a side and numbers in the other side)

$2x + 6x \leq 4 + 3$

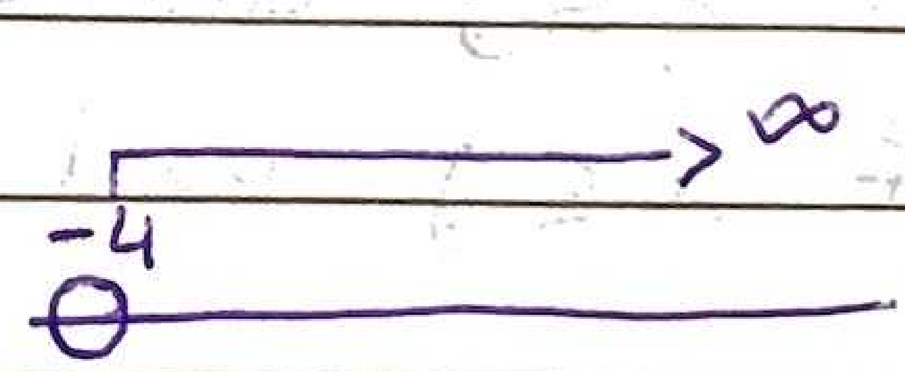
$8x \leq 7 \quad \div 8$

$x \leq \frac{7}{8}$



• ∴ S.S is $(-\infty, \frac{7}{8}]$

3. $-3x - 8 < 4$



$-3x < 4 + 8$

$-3x < 12 \quad (\div -3)$ ($<$) will be ($>$) because we divide both sides by (−)

$x > \frac{-12}{3}$

$x > -4$

S.S $(-5, \infty) = \{x \in \mathbb{R} : x > -4\}$

note　　and $\cap$　　or $\cup$　　6　　2

4. $x+5 \leq 3-5x < 2+7x$

$x+5 \leq 3-5x$ and $3-5x < 2+7x$

$x+5 \leq 3-5x \quad \wedge \quad 3-5x < 2+7x$

$\Rightarrow x+5x \leq 3-5 \quad \wedge \quad -5x-7x < 2-3$

$(\div 6)$ $6x \leq -2 \quad \wedge \quad -12x \leq -1$ $\quad \div -12$

$x \leq \frac{-2}{6} \quad \wedge \quad x \geq \frac{1}{12}$

S.s is $(-\infty, -\frac{1}{3}]$ and $(\frac{1}{12}, \infty)$

So the S.s is $(-\infty, -\frac{1}{3}] \cap (\frac{1}{2}, \infty) = \phi$

5. $x-3 < x-6$

$\Rightarrow x-x < -6+3$

$0 < -3$ which is not true

$\therefore$ S.s is ϕ

6. $x-5 > x-8$

$x-x > -8+5$

$0 > -3$ which is true for all real number $x \in \mathbb{R}$

$\therefore$ So S.s is $\mathbb{R}$

Some Properties of Inequalities :-

① If $a.b > 0$ then **either** $a > 0$ and $b > 0$

; **or** $a < 0$ and $b < 0$

and

If, $a.b \geq 0$, then **either** $a \geq 0$ and $b \geq 0$

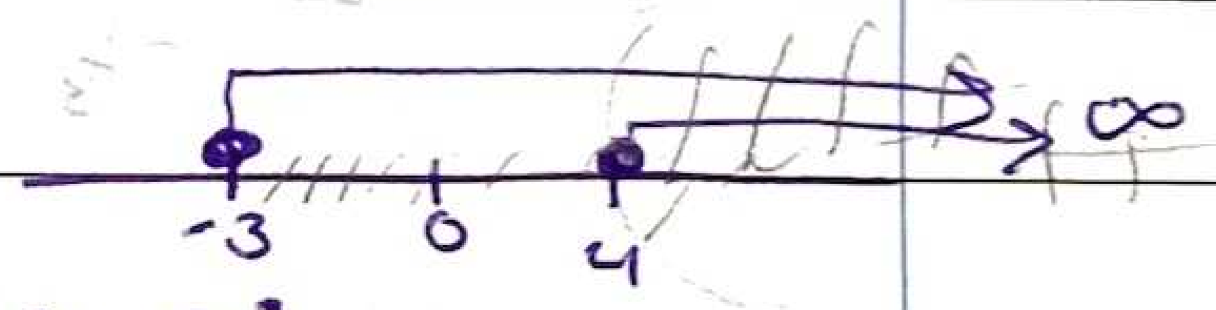
or $a \leq 0$ and $b \leq 0$

e.g $x^2 - x - 12 \geq 0$

$(x+3)(x-4) \geq 0$

ⓐ $(x+3) \geq 0$ and $(x-4) \geq 0$

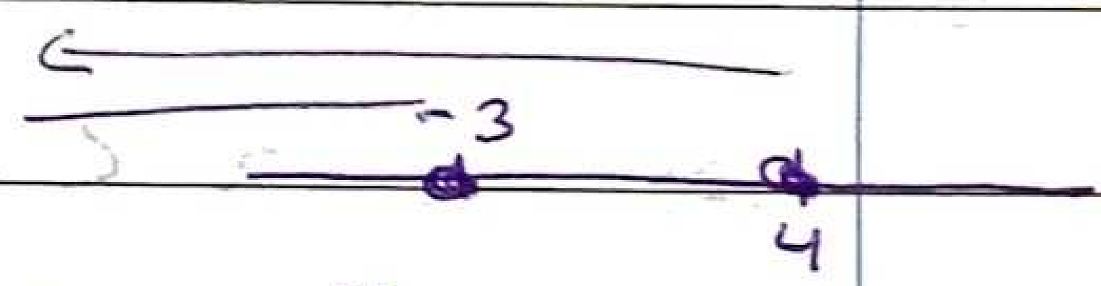
$x \geq -3 \quad \wedge \quad x \geq 4$



$[-3, \infty) \cap [4, \infty) = [4, \infty)$

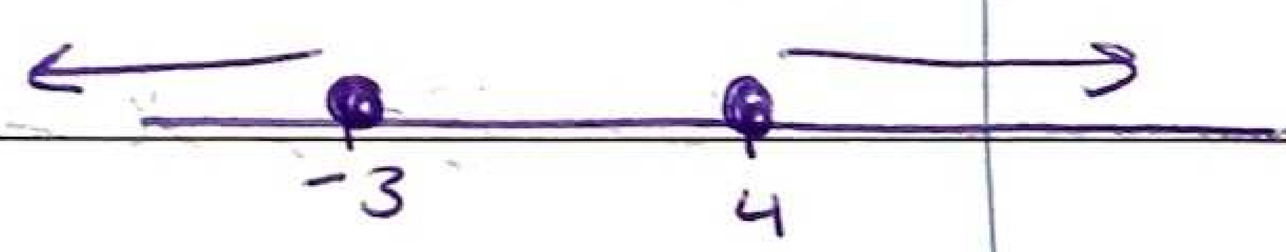
ⓑ $x+3 \leq 0$ and $x-4 \leq 0$

$x \leq -3 \quad \wedge \quad$ ~~and~~ $x \leq 4$



$(-\infty, -3] \cap (-\infty, 4] = (-\infty, -3]$

$\therefore$ S.s is $(-\infty, -3] \cup [4, \infty) = \mathbb{R} \setminus (-3, 4)$



H.W $(x-4)(x-3) > 0$

② If $a.b < 0$, then either $a > 0$ and $b < 0$
or $a < 0$ and $b > 0$

and

If $a.b \leq 0$, then either $a \geq 0$ and $b \leq 0$
or $a \leq 0$ and $b \geq 0$

e.g. $x^2 + 5x - 14 \leq 0$

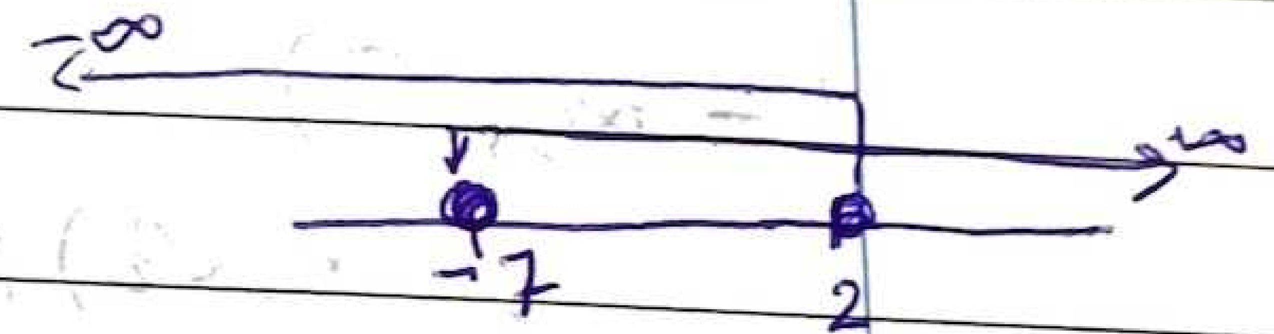
$(x+7)(x-2) \leq 0$

either

① $(x+7) \geq 0$ and $(x-2) \leq 0$

$x \geq -7$ $\wedge$ ~~x~~ $x \leq 2$

$[-7, \infty)$ $\wedge$ $(-\infty, 2]$



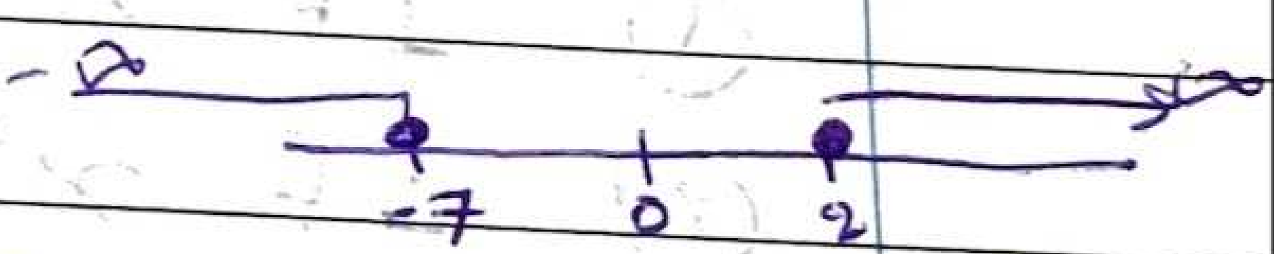
S.s is $[-7, \infty) \cap (-\infty, 2] = [-7, 2]$

or

② $(x+7) \leq 0$ and $(x-2) \geq 0$

~~$x+7 <$~~ $x \leq -7$ $\wedge$ $x \geq 2$

$(-\infty, -7]$ $\wedge$ $[2, \infty)$



S.s is $(-\infty, -7] \cap [2, \infty) = \phi$

$[-7, 2] \cup \phi = [-7, 2]$

H.W ① $(x+5)(x-4) \leq 0$

② $(x^2 - 5) < 0$

e.g $x^2 < 2$

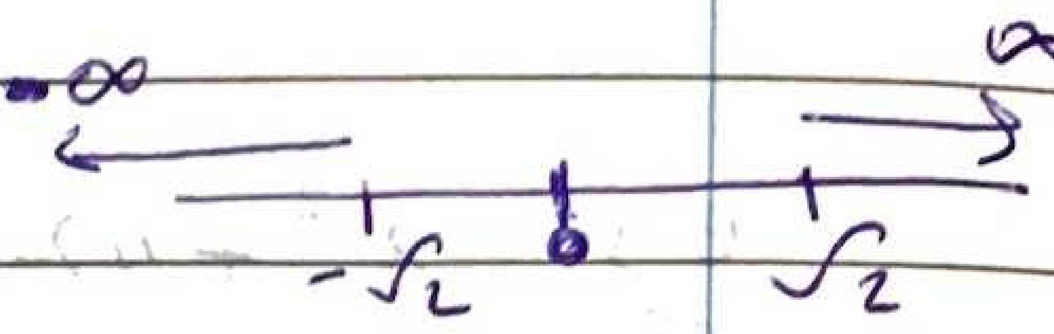
$x^2 - 2 < 0 \Rightarrow (x-\sqrt{2})(x+\sqrt{2}) < 0$

either

$(x-\sqrt{2}) > 0$ and $(x+\sqrt{2}) < 0$

$x > \sqrt{2}$ $\wedge$ $x < -\sqrt{2}$

$(\sqrt{2}, \infty)$ $\wedge$ $(-\infty, -\sqrt{2})$

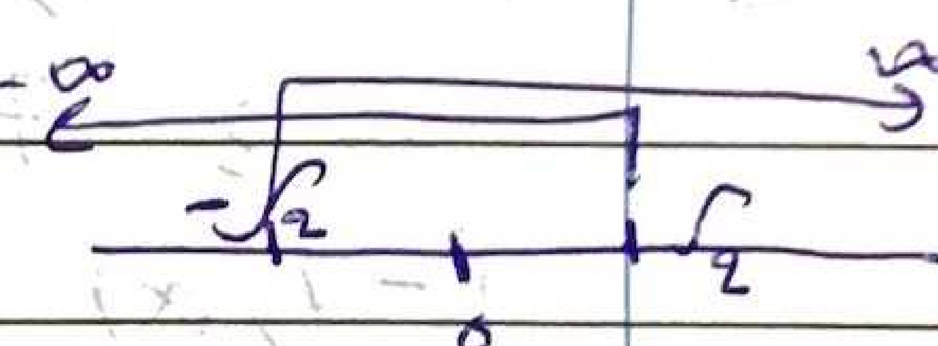


S.s is $(\sqrt{2}, \infty) \cap (-\infty, -\sqrt{2}) = \phi$

or $(x-\sqrt{2}) < 0$ and $x+\sqrt{2} > 0$

$x < \sqrt{2}$ $\wedge$ $x > -\sqrt{2}$

$(-\infty, \sqrt{2})$ $\wedge$ $(-\sqrt{2}, \infty)$



$(-\infty, \sqrt{2}) \cap (-\sqrt{2}, \infty) = (-\sqrt{2}, \sqrt{2})$

So the S.s is $\phi \cup (-\sqrt{2}, \sqrt{2}) = (-\sqrt{2}, \sqrt{2})$

Note

① If $x^2 < a \Rightarrow$ S.s is $(-\sqrt{a}, \sqrt{a})$

② If $x^2 \leq a \Rightarrow$ S.s is $[-\sqrt{a}, \sqrt{a}]$

③ If $x^2 > a \Rightarrow$ S.s is $\mathbb{R} \setminus [-\sqrt{a}, \sqrt{a}]$

④ If $x^2 \geq a \Rightarrow$ S.s is $\mathbb{R} \setminus (-\sqrt{a}, \sqrt{a})$

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③

Note ① If $x^2+a^2>0$

S.s is $\mathbb{R}$

e.g if $x^2+5>0 \Rightarrow$ S.s is $\mathbb{R}$

② If $x^2+a^2\leq 0$, $a>0$

S.s is ϕ

e.g

$x^2+7\leq 0 \Rightarrow$ S.s is ϕ

③ If $\frac{a}{b}>0$, then either $a>0$ and $b>0$

or $a<0$ and $b<0$

If $\frac{a}{b}\geq 0$, then either $a\geq 0$ and $b>0$

or $a\leq 0$ and $b<0$

If $\frac{a}{b}<0$, then either $a>0$ and $b<0$

or $a<0$ and $b>0$

If $\frac{a}{b}\leq 0$, then either $a\geq 0$ and $b<0$

or $a\leq 0$ and $b>0$

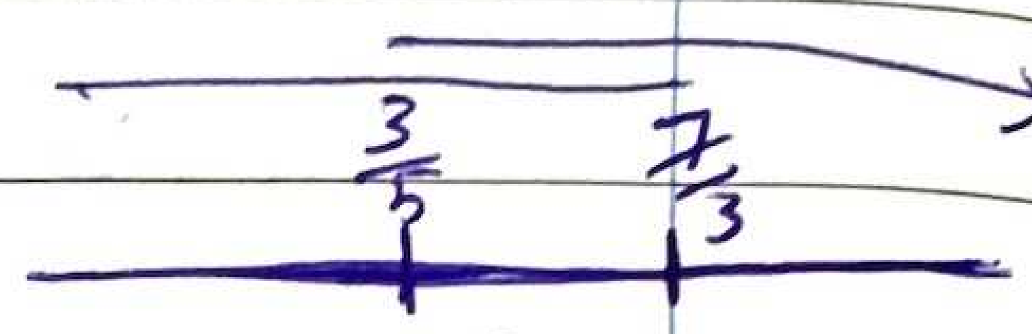
e.g. $\dfrac{5x-3}{7-3x} > 0$

either $5x-3>0$ and $7-3x>0$

$5x>3 \quad \wedge \quad -3x>-7 \quad \div(-3)$

$x>\frac{3}{5} \quad \wedge \quad x<\frac{7}{3}$

$(\frac{3}{5},\infty) \quad \wedge \quad (-\infty,\frac{7}{3})$



So the S.S. is $(\frac{3}{5},\infty)\cap(-\infty,\frac{7}{3}) = (\frac{3}{5},\frac{7}{3})$

or $5x-3<0$ and $7-3x<0$

$5x<3 \quad \wedge \quad -3x<-7 \quad \div(-3)$

$x<\frac{3}{5} \quad \wedge \quad x>\frac{7}{3}$

$(-\infty,\frac{3}{5}) \quad \wedge \quad (\frac{7}{3},\infty)$

The S.S is $(-\infty,\frac{3}{5})\cap(\frac{7}{3},\infty)=\phi$

So $(\frac{3}{5},\frac{7}{3})\cup\phi=(\frac{3}{5},\frac{7}{3})$

~~H.w~~ ~~3x+2~~

H.w / $\dfrac{3x+2}{1-4x} > 0$

e.g $\dfrac{3}{x-4} < 0$

either $3 > 0$ and $x - 4 < 0$

$\mathbb{R}$ $\wedge$ $x < 4$

$\mathbb{R}$ $\wedge$ $(-\infty, 4)$

S.s is $\mathbb{R} \cap (-\infty, 4) = (-\infty, 4)$

or $3 < 0$ and $x - 4 > 0$

ϕ $\wedge$ $x > 4$

ϕ $\wedge$ $(4, \infty)$

S.s is $\phi \cap (4, \infty) = \phi$

So the S.s is $(-\infty, 4) \cup \phi = (-\infty, 4)$

or

Since $3 > 0$ S.s is $\mathbb{R}$

$\Rightarrow x - 4 < 0 \Rightarrow x < 4$

S.s is $(-\infty, 4)$

another sol.

H.W ① $\dfrac{-x+3}{2x+5} \leq 0$

② $\dfrac{1}{3x-1} < 0$

Absolute Value

The absolute value of a number x, denoted by $|x|$, is defined by the formula

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

e.g Finding the absolute values

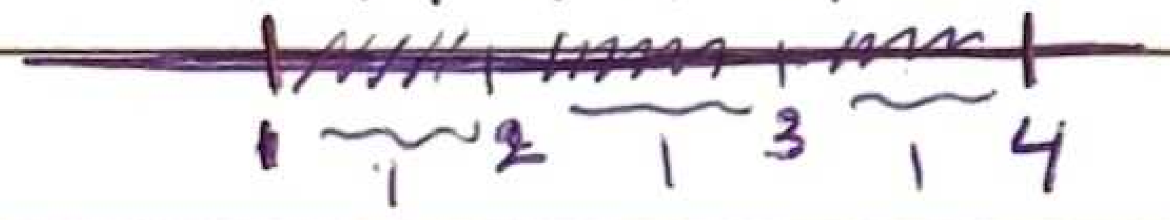
$|5| = 5$, $|0| = 0$, $|-3| = -(-3) = 3$

Geometrically:- the absolute value of x is the distance from x to 0 for every real number x, and $|x| = 0 \iff x = 0$, Also,

$|x-y|$ = the distance between x and y on the real line

$|3-(-5)| = 3+5 = 8$

$|4-1| = |1-4| = 3$ $|-5| = 5$ $|3| = 3$



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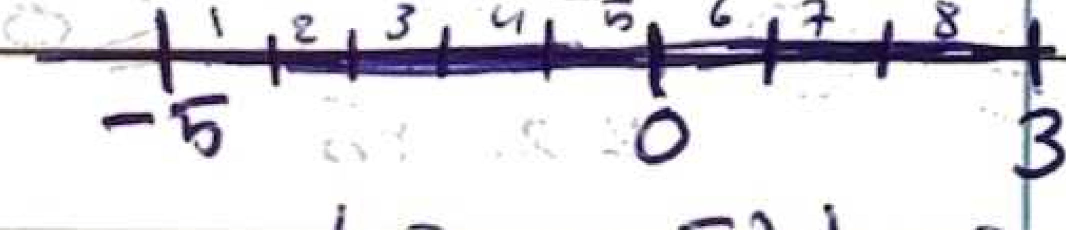


$|3-(-5)| = 8$

Absolute value Properties:-

① $|-x| = |x| \Rightarrow$ e.g $|-5| = 5$, $|5| = 5 \Rightarrow |-5| = |5|$

② $|x|^2 = x^2 \Rightarrow |-5|^2 = (5)^2 = 25$, $(-5)^2 = 25$

③ $|x \cdot y| = |x| \cdot |y| \Rightarrow$ e.g $|3 \cdot 4| = |12| = 12$ &

$|3| \cdot |4| = 3 \cdot 4 = 12$

④ $\left|\frac{x}{y}\right| = \frac{|x|}{|y|} \Rightarrow \left|\frac{12}{3}\right| = |4| = 4$ and

$\frac{|12|}{|3|} = \frac{12}{3} = 4$

5) $|x+y| \leq |x| + |y|$

e.g $|-4+8| = |4| = 4$ &

$|-4| + |8| = 4+8 = 12 \Rightarrow 4 < 8$

6) $|x-y| \geq |x| - |y|$

Absolute value and Intervals:-

If a is any positive number, then

1) $|x| = a \iff$ (if and only if) $x = \mp a$

2) $|x| < a \iff -a < x < a$

3) $|x| > a \iff x > a$ or $x < -a$

4) $|x| \leq a \iff -a \leq x \leq a$

5) $|x| \geq a \iff x \geq a$ or $x \leq -a$

e.g Solving an Equation with Absolute Values

1) $|2x - 3| = 7$

Sol By property 1) $2x - 3 = \mp 7$

$2x - 3 = 7$ and $2x - 3 = -7$

$\Rightarrow 2x = 7 + 3 \quad \wedge \quad 2x = -7 + 3$

$2x = 10 \quad \wedge \quad 2x = -4$

$x = 5 \quad \wedge \quad x = -2$

So the S.s of $|2x - 3| = 7$ is $x = 5$ & $x = -2$

(2) $|5-\frac{2}{x}|<1$

We have $|5-\frac{2}{x}|<1 \iff -1<5-\frac{2}{x}<1$ (property 2)

$-1-5<-\frac{2}{x}<1-5$

$\Rightarrow -6<-\frac{2}{x}<-4 \quad (\div -2)$

$3>\frac{1}{x}>2$

$\Rightarrow \frac{1}{3}<x<\frac{1}{2}$ $\therefore$ The S.S is $(\frac{1}{3},\frac{1}{2})$

(3) $|2x-3|\leq 1$ (property (4))

$\Rightarrow |2x-3|\leq 1 \Rightarrow -1\leq 2x-3\leq 1$

$\Rightarrow 3-1\leq 2x\leq 1+3$

$\Rightarrow 2\leq 2x\leq 4 \quad (\div 2)$

$1\leq x\leq 2$

$\therefore$ The S.S is $[1,2]$

(4) $|2x-3|\geq 1$ property (5)

$|2x-3|\geq 1 \Rightarrow 2x-3\geq 1$ or $2x-3\leq -1$

$\Rightarrow 2x-3\geq 1 \vee 2x-3\leq -1$

$2x\geq 1+3 \vee 2x\leq -1+3$

$\Rightarrow 2x\geq 4 \vee 2x\leq 2$

$x\geq 2 \vee x\leq 1$

$-\infty \leftarrow$ 1 2 $\rightarrow \infty$

$\Rightarrow$ The Solution set is $(-\infty,1]\cup[2,\infty)=\mathbb{R}\setminus(1,2)$

$|x| = 0 \iff x = 0$

⑤ $|1-8x| = 0 \Rightarrow$ cuz $\mp 0 = 0$

$1-8x = 0$

$\Rightarrow -8x = -1$

$\Rightarrow x = \frac{1}{8}$ So S.s is $x = \frac{1}{8}$

Note ① If $|x| = a$ where $a<0$, then

S.s is ϕ because $|x| = a$ has no Solution

e.g $|2x-4| = -5$

S.s is ϕ becuse absolute value of any number is positive

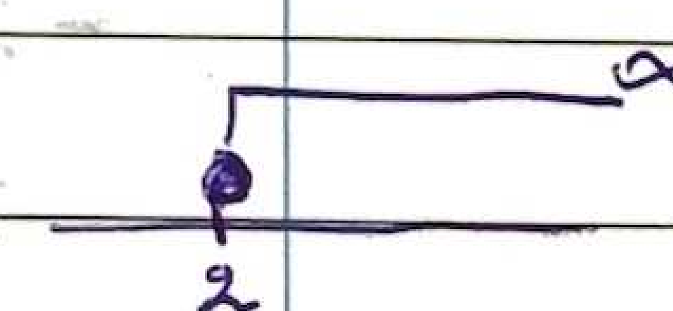
Note ② If $|x| = x$ it means that $x \geq 0$

e.g $|3x-6| = 3x-6$

$3x-6 \geq 0 \Rightarrow 3x \geq 6 \; (\div 3)$

$x \geq 2$

S.s is $[2, \infty)$



Note③ If $|x| = -x$

So S.s is ϕ

e.g $|2x-3| = 3-2x$

$\Rightarrow |2x-3| = -(2x-3)$

So S.s is ϕ

e.g Find the S.s of The following equations.

① $\frac{1}{2x-2} > 0$

either $1 > 0$ and $2x-2 > 0$

$\Rightarrow \mathbb{R} \cap \quad 2x > 2$

$\Rightarrow \mathbb{R} \cap \quad x > 1$

$\Rightarrow \mathbb{R} \cap \quad (1,\infty) = (1,\infty)$

or $\quad 1 < 0 \quad$ and $\quad 2x - 2 < 0$

$\phi \quad \cap \quad 2x < 2$

$\phi \quad \cap \quad x < 1$

S.S is $\phi \cap (-\infty,1) = \phi$

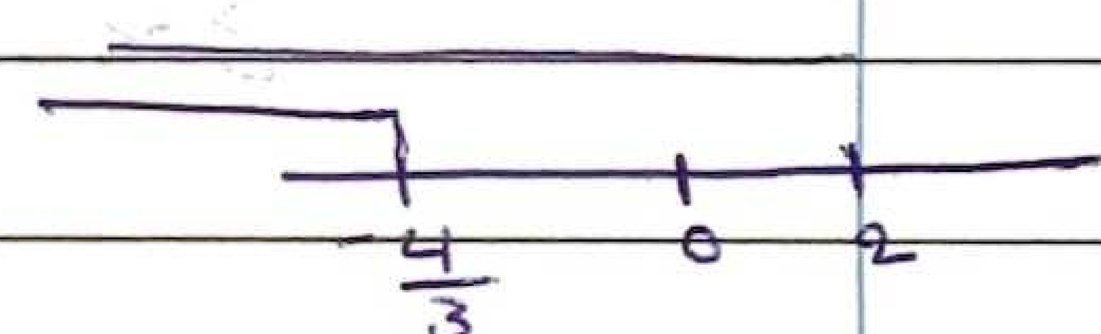
$\Rightarrow \phi \cup (1,\infty) = (1,\infty)$

e.g (2) $\dfrac{2-x}{3x+4} \leq 0$

$2 - x \geq 0 \quad$ and $\quad 3x+4 \leq 0$

$-x \geq -2 \quad \cap \quad 3x \leq -4$

$x \leq 2 \quad \cap \quad x \leq \frac{-4}{3}$



$\Rightarrow (-\infty, 2] \cap (-\infty, -\frac{4}{3}] = (-\infty, -\frac{4}{3}]$

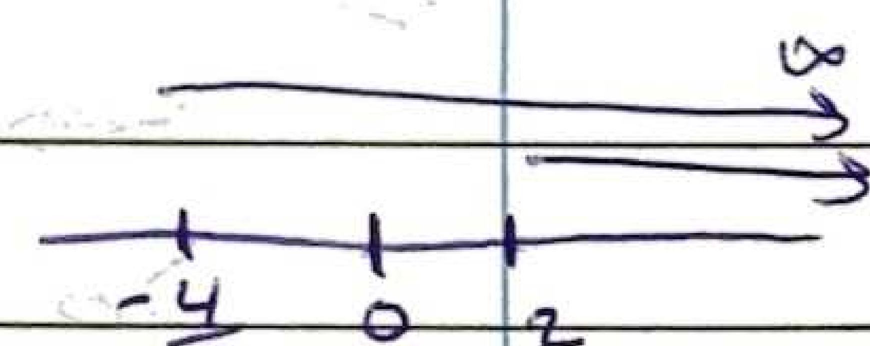
or

$2 - x \leq 0 \quad$ and $\quad 3x + 4 \geq 0$

$-x \leq -2 \quad \cap \quad 3x \geq -4$

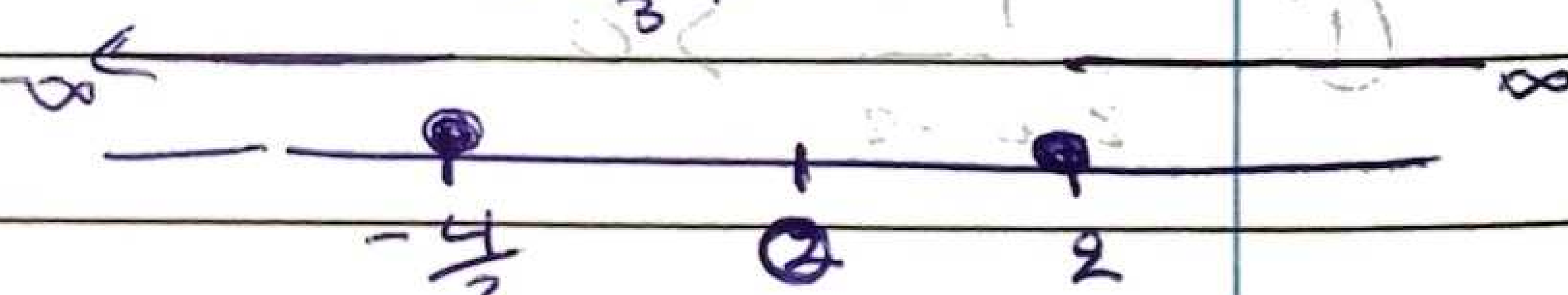
$x \geq 2 \quad \cap \quad x \geq \frac{-4}{3}$

$[2,\infty) \quad \cap \quad [-\frac{4}{3}, \infty)$



$= [2,\infty)$

$[-\infty, -\frac{4}{3}] \cup [2,\infty) = \mathbb{R} \setminus (-\frac{4}{3}, 2)$



H.W $|x+4| = 3x - 8$